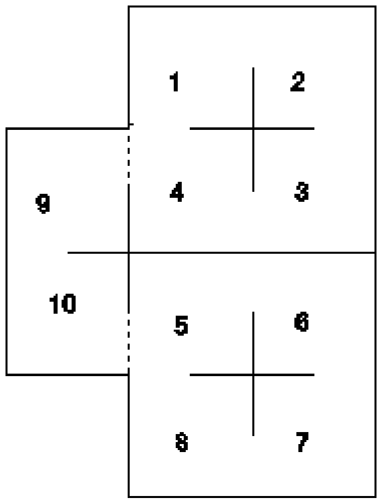
Ergotic Sets of States

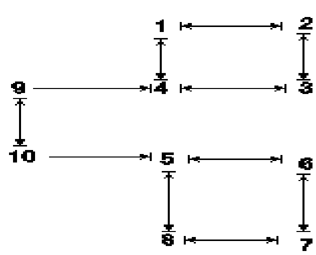
A set of states is *closed* if the set cannot be left. A closed set is *ergotic* if no proper subset is closed. For example, consider the following maze.



The dotted lines represent one-way doors. The mouse can go from room 9 to room 4 or from room 10 to room 5 but not visa versa.

There are two ergotic sets of states in this system: rooms 1, 2, 3 and 4; and rooms 5, 6, 7, and 8. Once the rat is in rooms 1 to 4 he can change rooms, but he can not leave this set of rooms. There are no absorbing states in this system, but the ergotic sets are like absorbing states; once a set is entered, it cannot be left. The other states are *transient*. These are states 9 and 10.

Below is a diagram of the relations between the states.



The sets of ergotic states can clearly be recognized as sets of states that cannot be left once entered. Assuming that the mouse has a probability of .50 of staying in the same room in a time period, the transition matrix is as follows:

room in a time period, the transition matrix is as follows:

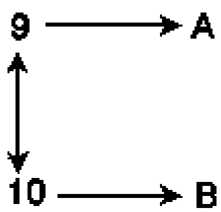
$$P := \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \qquad \text{ORIGIN} \equiv 1$$

Notice that this matrix can be partioned just like a matrix with two absorbing states. First come the two ergotic sets of states that do not lead to any other states. In the bottom left are two submatrices showing the probabilities of moving from the transient states to the ergotic states. This is like the submatrix R. Finally, in the lower right is a square submatrix, like Q, of the transitions among the transient states.

$$P = \left[\begin{array}{cccc|cccc|cc} 0.5 & 0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0.5 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0.5 & 0.25 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.25 & 0 & 0.25 & 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0.5 & 0.25 & 0 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.25 & 0.5 & 0.25 & 0 & 0 & 0 \end{array}\right.$$

0	0	0	0	0	0.25	0.5	0.25	0	0
0	0	0	0	0.25	0	0.25	0.5	0	0
0	0	0	0.25	0	0	0	0	0.5	0.25
0	0	0	0	0.25	0	0	0	0.25	0.5

Each ergotic set of states can be treated as one absorbing state. This is called the condensation of the full set of states. In the condensed graph, each ergotic set is represented by one point. Let's let A = {1, 2, 3, 4} and B = {5, 6, 7, 8}.



The corresponding p matrix has two absorbing states.

$$p := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

For this condensed system,

$$Q := \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix} \qquad R := \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

$$I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$N := (I - Q)^{-1}$$

$$N = \begin{bmatrix} 2.667 & 1.333 \\ 1.333 & 2.667 \end{bmatrix} \qquad N \cdot R = \begin{bmatrix} 0.667 & 0.333 \\ 0.333 & 0.667 \end{bmatrix}$$

These matrices have the usual interpretation. If the rat starts out in state 9, he has a 2/3 probability of ending up in set A of rooms and a 1/3 probability of ending up in set B. If he starts out in room 9, he will will on the average spend 2.667 time periods in room 9 and 1.333 time periods in room 10 before being absorbed into one of the ergotic sets.

Each ergotic state can be treated as an isolated system. The transition matrix for the sets A and B are as follows:

$i := 1..4 \qquad j := 1..4$

$A_{(i,j)} := P_{(i,j)}$

$B_{(i,j)} := P_{(i+4,j+4)}$

$$A = \begin{bmatrix} 0.5 & 0.25 & 0 & 0.25 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0.25 & 0 & 0.25 & 0.5 \end{bmatrix} \qquad B = \begin{bmatrix} 0.5 & 0.25 & 0 & 0.25 \\ 0.25 & 0.5 & 0.25 & 0 \\ 0 & 0.25 & 0.5 & 0.25 \\ 0.25 & 0 & 0.25 & 0.5 \end{bmatrix}$$

Each subsystem is regular and has an equilibrium distribution. These distributions describe what happens to the system once it enters an ergotic set of states.

$$A^{10} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \qquad B^{10} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

Now let's look at another example. Suppose that an organization has six divisions, two of them are involved in production, two in sales, and two in advertising. Ordinarily, new employees enter one of the advertising divisions in order to familiarize themselves with the companies products. Then they move into production or sales. The following transition matrix represents movements among divisions in this company.

$$P2 := \begin{bmatrix} .6 & .4 & 0 & 0 & 0 & 0 \\ .1 & .9 & 0 & 0 & 0 & 0 \\ 0 & 0 & .8 & .2 & 0 & 0 \\ 0 & 0 & .3 & .7 & 0 & 0 \\ .0 & .1 & .3 & .4 & .1 & .1 \\ .5 & .1 & 0 & .2 & .1 & .1 \end{bmatrix}$$

There are two ergotic sets of states, {1, 2} and {3, 4}. Combining divisions in the same ergotic set, we get the following transition matrix.

$$P2 := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ .1 & .7 & .1 & .1 \\ .6 & .2 & .1 & .1 \end{bmatrix}$$

Now, we can handle this transition matrix with two absorbing states in the usual way.

$$N2 := \left[I - \begin{bmatrix} .1 & .1 \\ .1 & .1 \end{bmatrix} \right]^{-1}$$

$$N2 \cdot \begin{bmatrix} .1 & .7 \\ .6 & .2 \end{bmatrix} = \begin{bmatrix} 0.187 & 0.812 \\ 0.687 & 0.312 \end{bmatrix}$$

This shows the probability of working in the production or sales divisions depending on one's first job. Within each of the ergotic sets of divisions we can also calculate an equilibrium distribution.

$$\begin{bmatrix} .6 & .4 \\ .1 & .9 \end{bmatrix}^{20} = \begin{bmatrix} 0.2 & 0.8 \\ 0.2 & 0.8 \end{bmatrix}$$

$$\begin{bmatrix} .8 & .2 \\ .3 & .7 \end{bmatrix}^{20} = \begin{bmatrix} 0.6 & 0.4 \\ 0.6 & 0.4 \end{bmatrix}$$

Chapter 9 Homework
Ergotic States

1. In each of the following systems, identity the absorbing states, ergotic sets of states, and transient states. You will have to rearrange the rows and columns. Put the ergotic sets and the absorbing states first.

a.

$$P1 := \begin{bmatrix} .6 & 0 & .4 \\ .5 & .1 & .4 \\ .1 & 0 & .9 \end{bmatrix}$$

b.

$$P2 := \begin{bmatrix} .6 & 0 & .3 & .1 & 0 \\ 0 & .8 & 0 & .2 & 0 \\ .4 & .2 & 0 & .2 & .2 \\ 0 & .3 & 0 & .7 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

c.

$$P3 := \begin{bmatrix} .5 & 0 & .4 & .1 & 0 & 0 \\ .5 & 0 & .3 & .1 & .1 & 0 \\ .3 & 0 & .3 & .4 & 0 & 0 \\ .1 & 0 & .2 & .7 & 0 & 0 \\ 0 & .1 & 0 & 0 & 0 & .9 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Find the equilibrium distribution for each ergotic set of states.
3. For each of the matrices in problem 1, write down a new transition matrix in which each set of ergotic states are treated as a single state. What are the absorbing states of this new transition matrix? If there is more than one absorbing state, find the probabilities of being absorbed by each one of them.