Chapter 5
Addition and Multiplication of Matrices

Matrices can be added and multiplied according to their own special rules. If A and B are two matrices, then A+B is their sum and AB is their product. For example, consider the following two matrices giving the amount of trade between countries in two different years.

$$T1 := \begin{bmatrix} 0 & 50 & 70 & 80 & 30 \\ 20 & 0 & 40 & 30 & 50 \\ 60 & 70 & 0 & 60 & 80 \\ 70 & 20 & 50 & 0 & 10 \\ 10 & 60 & 70 & 5 & 0 \end{bmatrix} \qquad T2 := \begin{bmatrix} 0 & 59 & 70 & 80 & 30 \\ 21 & 0 & 40 & 30 & 50 \\ 63 & 66 & 0 & 60 & 80 \\ 68 & 24 & 50 & 0 & 10 \\ 17 & 70 & 90 & 15 & 0 \end{bmatrix}$$

ADDITION OF MATRICES

For addition of matrices, the dimensions of the two must be exactly the same or else addition is not defined. T1+T2 is a new 5 by 5 matrix whose elements are the sums of the corresponding elements of T1 and T2: $(T1 + T2)_{ij} = T1_{ij} + T2_{ij}$

$$T1 + T2 = \begin{bmatrix} 0 & 109 & 140 & 160 & 60 \\ 41 & 0 & 80 & 60 & 100 \\ 123 & 136 & 0 & 120 & 160 \\ 138 & 44 & 100 & 0 & 20 \\ 27 & 130 & 160 & 20 & 0 \end{bmatrix}$$

Let us look again at the matrices of liking and helping that we used in the last chapter.

P := READPRN("rdpos.prn"H := READPRN("rdhlp.prn")

$$P + H = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \end{bmatrix}$$

The matrix P + H tells us how many relations there were between pairs of individuals. If a cell is zero, it tells us that that the pair were not friends nor did one help the other. A one tells us that they were either friends or one helped the other. A two tells us that they were both friends and one helped the other. In other words, P + H tells us something about the strength of the bonds between pairs of people.

The addition of matrices has two abstract properties shared with the arithmetic addition of numbers. First, addition of matrices, like the addition of numbers, is ***commutative***. This means that for any two matrices, A + B = B + A. Note that not all arithmetic operations are commutative. For example, neither subtraction nor division is commutative; a - b is not in general equal to b - a, and a/b is usually not equal to b/a.

The next property, *associativity*, is more subtle. For numbers, we know that (a + b) + c = a + (b + c). This means that if we first add a and b and add the result to c we get the same number as when we add a to the sum of b and c. The same is true for matrix addition: it is always true that (A + B) + C = A + (B + C).

MULTIPLICATION OF MATRICES

Suppose that we have two vectors, p and q, representing the price of a set of goods and the quantity that was sold.

$p := (3.43\ \ .99\ \ 17.49\ \ .34\ \ 2.23)$

$$q := \begin{bmatrix} 5 \\ 23 \\ 1 \\ 35 \\ 9 \end{bmatrix}$$

It's important that p is a row vector and q a column vector.   p is also "a one by five matrix and q is a five by one matrix. How would we go about calculating the total amount that was spent? We would do this by multiplying the price of each good times the quantity of that good that was bought and then adding up the result.

OF

$$\sum_i p_{1,i} \cdot q_{i,1} = 89.38$$

This is exactly how the product of these two matrices is defined. The product is the sum of the products of the corresponding elements.

$p \cdot q = (89.38)$

Now let us make q into a five by two matrix in which the two columns represent the amounts purchased by two different shoppers.

$$q := \begin{bmatrix} 5 & 0 \\ 23 & 35 \\ 1 & 0 \\ 35 & 46 \\ 9 & 5 \end{bmatrix}$$

We would calculate the amount spend by each of the two shoppers by multiplying the price matrix times the two columns of the q matrix.

$$\sum_i p_{1,i} \cdot q_{i,1} = 89.38$$

For the amount spent by the first shopper.

$$\sum_i p_{1,i} \cdot q_{i,2} = 61.44$$

For the amount spent by the second shopper.

This is exactly how the product of p and q is defined.

$$p \cdot q = (89.38 \quad 61.44)$$

Now let us change the problem again. Let p be the prices at two different stores. p is now a two by five matrix.

$$p := \begin{bmatrix} 3.43 & .99 & 17.49 & .34 & 2.23 \\ 2.43 & 1.07 & 14.33 & .53 & 1.99 \end{bmatrix}$$

Now we want to ask how much each shopper would spend at each of the two stores. To do this we would multiply each of the store rows from p times each of the shopper columns from q.

STORE 1, SHOPPER 1

$$\sum_i p_{1,i} \cdot q_{i,1} = 89.38$$

STORE 1, SHOPPER 2

$$\sum_i p_{1,i} \cdot q_{i,2} = 61.44$$

STORE 2, SHOPPER 1

$$\sum_i p_{2,i} \cdot q_{i,1} = 87.55$$

STORE 2, SHOPPER 2

$$\sum_i p_{2,i} \cdot q_{i,2} = 71.78$$

Shopper 1 is better off at store 2. Shopper 2 is better off at store 1. The product of matrices p and q is defined in exactly the same way.

$$p \cdot q = \begin{bmatrix} 89.38 & 61.44 \\ 87.55 & 71.78 \end{bmatrix}$$

More generally, let A be a matrix with m rows and n columns and let B be a matrix with n rows and p columns. **Note that the requirement for matrix multiplication is that the number of columns of the first matrix equals the number of rows of the second.** Then AB is a matrix with m rows and p columns formed by multiplying the corresponding elements of each row of A and each column of B and adding up the products. Let C = AB. Then the elements in the product are defined in the following way.

$$C = AB$$

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots$$

Now let us work out an example.

$$A := \begin{bmatrix} 5 & 3 & 2 & 0 \\ 9 & 4 & 8 & 4 \end{bmatrix} \qquad B := \begin{bmatrix} 2 & 0 & 1 \\ 9 & 0 & 3 \\ 1 & 4 & 1 \\ 4 & 2 & 7 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 39 & 8 & 16 \\ 78 & 40 & 57 \end{bmatrix}$$

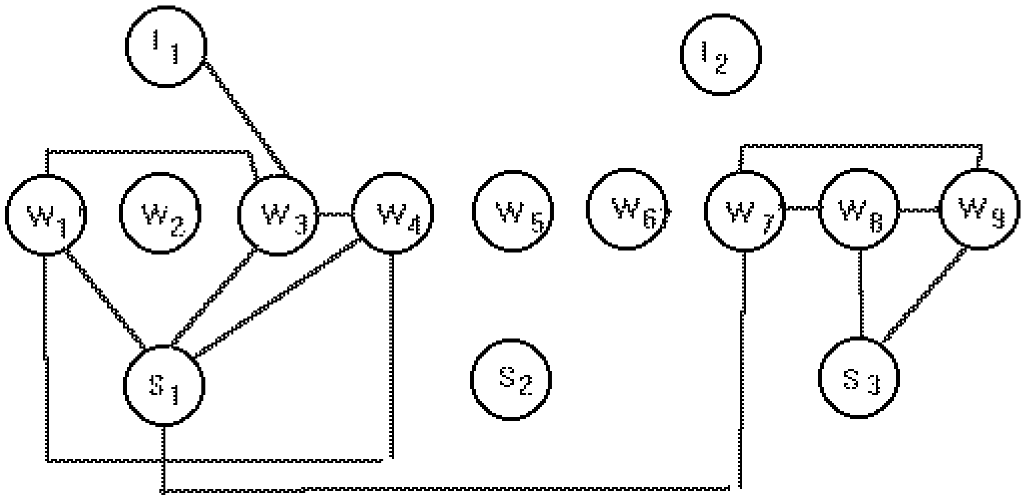
Surprisingly, matrix multiplication is not commutative. One reason is that sometimes AB is defined but BA is not. For example, BA is not defined for the two matrices above. Even if AB and BA are both defined, usually, AB and BA are not equal. We will see examples of this in later chapters.

However, matrix multiplication is associative; (AB)C = A(BC) is always true if the appropriate multiplications are defined.

MULTIPLICATION OF INCIDENCE MATRICES

Now let us look at a diagram that corresponds to the P matrix of friendship choices in the Bank Wiring Room.

Figure 1



The incidence matrix for this graph is the 14 by 14 matrix P. $P_{ij} = 1$ only if i and j are friends. If P is the relationship as well as the matrix, then $P_{ij} = 1$ only if iPj. Now let's look at a different relationship, P^2, having friends in common. iP^2j only when i and j have a common friend; they may or may not be friends with each other. More specifically, iP^2j if and only if there is some person k such that iPk and kPj. Person k is the friend they have in common. Why we call this relationship P^2 will be clearer soon.

Looking at the diagram, we can see that W_1 and W_3 have friends in common (two of them), while W_1 and S_3 do not. We could graph the relationship P^2. In this graph there would be a line between I_1 and W_1 even though they are are not themselves friends. Similarly, there would not be a line between I_1 and W_3 because although they are friends they do not share a friend.

We could also create an incidence matrix for this relationship. What I want to show is that the incidence matrix for the relation P^2 is (almost) equal to the matrix P^2 = P times P. Consider again the definition of the relationship P^2: iP^2j if and only if there exists a person k such that iPk and kPj. Now consider the definition of the i,j element in the matrix P^2 = P times P.

$$\sum_k P_{i,k} \cdot P_{k,j}$$

This will be the sum of the number of instances in which i and k are friends and k and j are friends. **Thus, P^2_{ij} is just the number of common friends k of both i and j.** P^2 is not quite the incidence matrix for the relationship *P^2* because it can contain elements greater than one (when two people have more than one friend in common). But it can easily be converted into an incidence matrix by changing all values that are greater than one into one.

$$P^2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 0 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 4 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 & 2 & 3 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 3 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 2 \end{bmatrix}$$

The order of the rows and columns in P^2 and P is I_1, I_2, W_1, W_2, W_3, W_4, W_5, W_6, W_7, W_8, W_9, S_1, S_2, S_3. $P^2_{53} = 2$ because 5 (W3) and 3 (W1) have two friends in common, W4 and S1). The numbers on the main diagonal have a different interpretation. It's not too difficult to show that they are simply the number of friends each person has.

$(P^2)_{ii} = \Sigma_k P_{ik}P_{ki}$ matrix multiplication rule

$= \Sigma_k P_{ik}P_{ik}$ P is symmetric, so $P_{ki} = P_{ik}$

$= \Sigma_k P^2_{ik}$

$= \Sigma_k P_{ik}$ because P consists of ones and zeros, and $0^2 = 0$, $1^2 = 1$

Therefore, the main diagonal of the square of the symmetric matrix P shows the number of friends of each person.

The interpretation of the elements of R^2 is subtly different when *R* is not a completely symmetric relationship. Consider the incidence matrix R for any relation *R*. $(R^2)_{ij}$ is the number of paths of length 2 from position i to position j. A path of length 2 from i to j means that there is at least one person k such that iRk and kRj. But then $R_{ik} = R_{kj} = R_{ik}R_{kj} = 1$ for each such intermediate person k. So, $(R^2)_{ij}$ is the number of paths of length 2 from i to j.

The most general statement is as follows:

Let R be the incidence matrix for a relationship *R*, which may or may not be symmetric. Then, $(R^n)_{ij}$ is the number of paths of length n from person i to person j,

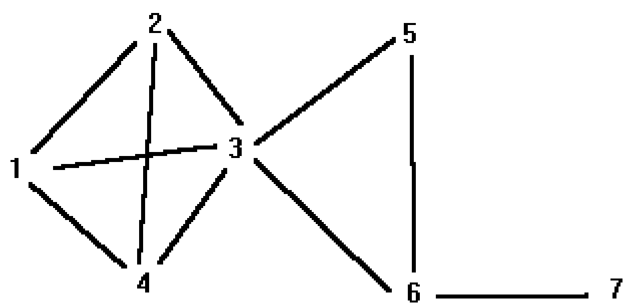
LOCATING CLIQUES

Cliques are sets of friends. "Cliquishness" means exclusivity; members of a clique are not friends with those outside the clique. How strong and exclusive does a set of friends have to be before they form a clique. There is no definite and authoritative answer to this question. In this section we will explore some possible definitions of cliques.

Definition: Let R be a symmetric and reflexive relationship. A *strong clique* S is a set of people for whom aRb for all a and b in S and S is not a proper subset of some larger strong clique. [Note: A set S is a proper subset of a set T if all elements of S are in T and there is at least one element of T that is not in S.]

There are two parts to this definition, and both parts are important. Consider the following hypothetical graph of friendship relations among a group of people.

Figure 2



The three strong cliques are {1, 2, 3, 4}, {3, 5, 6}, and {6, 7}. {1, 2, 3} is not a strong clique because it forms a subset of the larger strong clique {1, 2, 3, 4}. Notice that person 3 belongs to two different strong cliques.

A *weak clique*, on the other hand, is a set of people who are connected by ties of friendship even if they are very indirect. Two people are in the same weak clique if they are friends, or if they share friends, or if their friends are friends, and so forth. In terms of graphs, two people, by this definition, are in the same clique if there is a path of any length connecting them.

Definition: Let R be a symmetric reflexive relation. A weak clique is a set of people all of whom are connected directly or indirectly by paths of any length and who are not a subset of a larger weak clique.

In the above diagram there is one weak clique consisting of all seven individuals. Individuals 1, 2, 3, and 4 do not form a weak clique because they are a subset of a larger weak clique, the set of all seven individuals.

One mathematical consequence of the definition of weak cliques is that being in the same clique is an equivalence relation. That is to say, it is symmetric, reflexive, and transitive.

Claim. Let aWb mean that there is a path of friendship choices from a to b. If friendship R is symmetric and reflexive, then W (belonging to the same weak clique) is an equivalence relation.

W is reflexive because R is reflexive. Similarly, since R is symmetric, if there is a path from a to b, there also exists a path in the opposite direction from b to a (if R is not symmetric, then W need not be symmetric). Finally, W is transitive. Remember that W is transitive is whenever aWb and bWc then aWc. Suppose there is a path in a graph from a to b and a path from b to c. Must there be a path from a to c? Of course! There must be at least one path from a to c, one that passes through b. There may be others as well.

Any equivalence relation breaks a set up into equivalence classes, separate sets of individuals who are not related to each other. Look at the diagram of friendships in the bank wiring room. There is one large weak clique with nine members and five smaller ones consisting of just one person. These five people are "isolates."

There is a way to detect weak cliques using matrix multiplication. Remember that R to the power k gives the number of paths of length k connecting every pair of persons. If there are n people in a group the longest non-redundant path (a path that does not backtrack on itself and does not pass through any person more than once) can be at most of length n-1. This does not mean, however, that a path of length n-1 will exist or that it must be non-redundant even if it does. For example, among the fourteen individuals in the Bank Wiring Room, the longest possible non-redundant path would pass through each person once and would be of length thirteen. In the Bank Wiring Room, such a path does not exist since some workers are isolates and are not connected to anyone else. But, if we added up the first n-1 powers of the matrix, we would get the total number of paths connecting each pair. By summing up the first n-1 powers we are guaranteed of not missing any paths. If this number is zero, the pair are unconnected and must belong to different weak cliques. If it is greater than zero, they must belong to the same weak clique.

Let's look at the seven-person group described by Figure 2. We could see by eye that they formed one weak clique. This can also be demonstrated by looking at powers of the matrix of their relationships.

$$P := \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

We need only look at paths up to length six; if two positio aren't connected by a path of length 6, they aren't connected at all. So, let's take the sum of the first six powers of the matr P.

$$k := 1 .. 6$$

$$\sum_{k=1}^{6} P^k = \begin{bmatrix} 336 & 336 & 422 & 336 & 188 & 202 & 58 \\ 336 & 336 & 422 & 336 & 188 & 202 & 58 \\ 422 & 422 & 554 & 422 & 248 & 264 & 86 \\ 336 & 336 & 422 & 336 & 188 & 202 & 58 \\ 188 & 188 & 248 & 188 & 122 & 132 & 44 \\ 202 & 202 & 264 & 202 & 132 & 150 & 46 \\ 58 & 58 & 86 & 58 & 44 & 46 & 20 \end{bmatrix}$$

As we can see, there are paths connecting all pairs of points, so they are in one weak clique.

FURTHER READING

The best single source for the description of the different types of cliques is the following manual for a computerized network analysis program.

Steve Borgatti, Martin Everett, and Linton Freeman, UCINET IV Version 1.0 Reference Manual. Columbia, S.C.: Analytic Technologies. 1992.

CHAPTER 5 HOMEWORK

ADDITION AND MULTPLICATION OF MATRICES

1. Add the following pairs of matrices, if possible. If not possible, explain why.

a. $\begin{bmatrix} 7 & -4 \\ 2 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 6 & 2 \end{bmatrix}$

b $\begin{bmatrix} 4 & 3 \\ -1 & 4 \\ -5 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 6 & -4 \\ 7 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 1 \\ 9 & 7 & 2 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 1 \\ 9 & 7 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 7 & -1 \\ 5 & 2 & -2 \\ 3 & 8 & -3 \end{bmatrix}$

e. $\begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix} + \begin{bmatrix} -3 \\ 6 \\ 1 \end{bmatrix}$

f. $(2 \;\; 8) + (6 \;\; 1)$

g. $\begin{bmatrix} 5 & -3 & -1 \\ 9 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

h. $\begin{bmatrix} 5 & 1 \\ 6 & 1 \end{bmatrix} + 7$

2. Multiply the following matrices, where this is possible. If not possible, explain why.

a. $\begin{bmatrix} 4 & 8 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 6 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 7 & -4 \\ 2 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 6 & 2 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 3 \\ -1 & 4 \\ -5 & 7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 6 & -4 \\ 7 & 2 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 3 & 4 \\ 5 & 2 & 1 \\ 9 & 7 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 7 & -1 \\ 5 & 2 & -2 \\ 3 & 8 & -3 \end{bmatrix}$

e. $\begin{bmatrix} 7 & 1 & 1 \\ 2 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 6 & 1 & 3 \\ 0 & 2 & 2 & 6 \\ 5 & 1 & 8 & 4 \end{bmatrix}$

f. $\begin{bmatrix} 1 & 1 & 8 & 6 \\ 6 & 9 & 2 & 3 \\ 8 & 6 & 4 & -7 \\ 2 & 5 & 1 & -7 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

g. $\begin{bmatrix} 4 \\ 7 \\ 3 \end{bmatrix} \cdot (2 \;\; 4 \;\; 1)$

3. Find ther weak and strong cliques in the following structure.

$$P := \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$