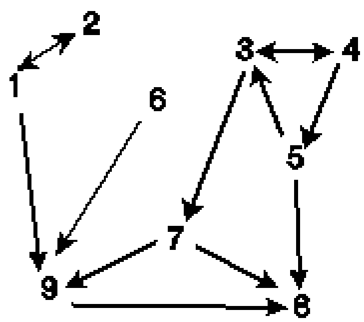
CHAPTER 10
DYNAMIC MODELS OF INFLUENCE

Suppose we were to ask the members of a group about which other group members influenced them the most. In this graph, a line from one member to another means that the first influenced the second.



We could also put this in the form of a matrix P.

$$R := \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

Let's suppose that people have different degrees of influence over each other. Suppose that person 1 influences person 2 more than visa versa. Specifically, suppose that when person 1 tends to move 20% of the way to person 2's opinion and that person 2 moves 40% of the way toward person 1 whenever they express an opinion. Suppose that they both see a new movie. On a 10 point scale, 1 likes it a 9 and 2 likes it a 3. After they talk, 1 moves 20% of the way toward 2 and 2 moves 40% of the way toward 1. So, their new opinions are:

1's new opinion $\quad 9 - .2\cdot(9 - 3) = 7.8$

2's new opinion $\quad 3 + .4\cdot(9 - 3) = 5.4$

This can also be expressed by matrix multiplication. P will be a two by two matrix showing how much each person influences others. A will be a vector of initial opinions.

$$P := \begin{bmatrix} .8 & .4 \\ .2 & .6 \end{bmatrix} \qquad A := \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

The sum of each column is 1.00 The main diagonal terms simply indicate the degree to which each person's opinions are not influenced by others. The other terms indicate the amount of influence people have an others' opinions. P_{ij} means the amount of influence i has on j's opinion. P_{12} = .4 because when 1 and 2 interact, 2 moves 40% of the way to i's opinion. The following matrix equation then gives the new opinions as a function of the old opinions and the influence structure, the matrix P

$$P^T \cdot A = \begin{bmatrix} 7.8 \\ 5.4 \end{bmatrix}$$

The same logic implies that the next time they discuss the movie, their opinions will be even closer to each other.

$$P^T \cdot \left(P^T \cdot A\right) = \begin{bmatrix} 7.32 \\ 6.36 \end{bmatrix}$$

This process can be continued until they reach a consensus after a larger number of discussions.

$$\left(P^T\right)^{10} \cdot A = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

Now let's go back to the group of nine. Suppose that we discover the following pattern of influence. Note that the sum of each column is 1.00, as with the previous P.

$$P := \begin{bmatrix} .8 & .4 & 0 & 0 & 0 & 0 & 0 & 0 & .2 \\ .2 & .6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .3 & .5 & 0 & 0 & .7 & 0 & 0 \\ 0 & 0 & .35 & .5 & .8 & 0 & 0 & 0 & 0 \\ 0 & 0 & .35 & 0 & .2 & 0 & 0 & .3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & .5 \\ 0 & 0 & 0 & 0 & 0 & 0 & .3 & .3 & .1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & .3 & .2 \end{bmatrix}$$

We are going to be asking the following questions. Will this group reach a consensus? If the whole group does not reach a consensus, are there subgroups that will come to agree? Which positions are the most influential?

Influence Structures as Markov Matrices

Notice that the transpose of an influence matrix P is a Markov chain transition matrix because each row of P^T sums to 1.00. This influence process is not a Markov chain; probabilities are involved at all. But because each matrix has only positive values and the rows sum to one, we can use some of the same techniques.

1. A regular matrix was one in which all values were strictly positive for some power. High powers of the regular matrix approach a matrix in which all rows are equal.

If P^T is regular, then $(P^T)^k$ approaches a matrix in which every row is equal to an equilibrium vector w. Since $(P^T)^k$ shows mutual influences after k time periods, this means that everyone's attitude approaches the same combination of their initial attitudes. The group reaches consensus. Person i's effect on the consensus is given by w_i in the equilibrium vector.

For example, consider persons 3, 4, and 5 in isolation. Their mutual influence is described by the following matrix S

$i := 1..3$ $\qquad$ $j := 1..3$ $\qquad$ $ORIGIN \equiv 1$

$$S_{(i,j)} := P_{(i+2,j+2)}$$

$$S = \begin{bmatrix} 0.3 & 0.5 & 0 \\ 0.35 & 0.5 & 0.8 \\ 0.35 & 0 & 0.2 \end{bmatrix}$$

$$\left(S^T\right)^{10} = \begin{bmatrix} 0.352 & 0.493 & 0.154 \\ 0.352 & 0.493 & 0.154 \\ 0.352 & 0.493 & 0.154 \end{bmatrix}$$

This means that their final attitudes are most strongly determined by 4 and that 5 has the least influence. Suppose that 2 gave a movie a "10" while 1 and 3 gave it a "0." The final consensus would be:

$$\left(S^T\right)^{10} \cdot \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.934 \\ 4.934 \\ 4.934 \end{bmatrix}$$

Although outnumbered 2 to 1, the final group attitude is half the way between 2's attitude and the attitudes of 1 and 3.

2. An ergotic set of states can be entered but cannot be left.

The matrix P^T tells us who influences whom. Ergotic sets of states correspond to sets of positions that are not influenced by anyone else. We will call these *power sets*. Since they are not influenced by anyone else, the relative power within each power set can be calculated in isolation.

Position 6 is an power set by itself. We have already calculated that the relative power of the three positions in the power set {3, 4, 5} is .352, .493, .154. We can calculate the relative power in the power set {1, 2} by looking at high powers of their submatrix. Position 1 has twice as much influence as position 2.

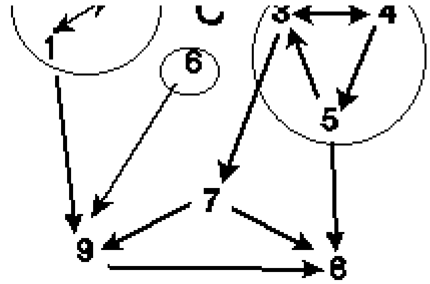
$$P := \begin{bmatrix} .8 & .4 \\ .2 & .6 \end{bmatrix}$$

$$\left(P^T\right)^{10} = \begin{bmatrix} 0.667 & 0.333 \\ 0.667 & 0.333 \end{bmatrix}$$

3. Each ergotic set of states can be replaced by a single absorbing state. One can then calculate the probabilities that the system will be absorbed by its ergotic sets of states.

Similarly, each power set of positions can be replaced by a single position, and the ultimate influence of each power set can be calculated.

Letting A = {1, 2} , B = {3, 4, 5}, and C = {6}, our new graph is:



The matrix corresponding to this reduced graph is:

$$P := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & .2 \\ 0 & 1 & 0 & .7 & .3 & 0 \\ 0 & 0 & 1 & 0 & 0 & .5 \\ 0 & 0 & 0 & .3 & .3 & 0 \\ 0 & 0 & 0 & 0 & .1 & .1 \\ 0 & 0 & 0 & 0 & .3 & .2 \end{bmatrix}$$

P^T is a Markov matrix.

$$P^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.3 & 0 & 0 \\ 0 & 0.3 & 0 & 0.3 & 0.1 & 0.3 \\ 0.2 & 0 & 0.5 & 0 & 0.1 & 0.2 \end{bmatrix}$$

Since P^T is a Markov matrix, we can partition it into submatrices. In the upper left is an identity matrix. In the lower right is a three by three matrix Q showing influence relations among the positions that are not part of power set. $N = (I-Q)^{-1}$ shows the power relations among these positions. R shows the direct power relations between the non-power positions and the power sets. NR shows the total degree to which each non-power position is dependent on each power set.

$$Q_{(i,j)} := \left(P^T\right)_{(i+3,\, j+3)}$$

$$\left[\left[\begin{matrix} 1 & 0 & 0 \end{matrix}\right] \qquad \right]^{-1}$$

$$N := \left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - Q \right]$$

$$R := \begin{bmatrix} 0 & .7 & 0 \\ 0 & .3 & 0 \\ .2 & 0 & .5 \end{bmatrix}$$

$$N \cdot R = \begin{bmatrix} 0 & 1 & 0 \\ 0.087 & 0.696 & 0.217 \\ 0.261 & 0.087 & 0.652 \end{bmatrix}$$

The rows of this matrix correspond to the three non-power positions 7, 8, and 9, the columns to the three power sets A, B, and C. As you can see from the graph, position 7 is influenced only by power set B. The consensus within power set B is determined mostly by positions 3 and 4. So, 7's opinions are determined mostly by positions 3 and 4, even though there is no direct connection from position 4 to position 7. Position 8's opinions are also mostly determined by set B because his opinions are affected by position 5, a member of set B, and position 7, whose opinions are also determined largely by set B. Finally, position nine is affected mostly by power set C (position 6). Power set B exerts a small effect even though there is no direct connection between set B and position 9.

Therefore, we can draw the following conclusions. Assuming that the Markov matrix for each power set is regular, so that each arrives at a consensus,

1. Each person's ultimate opinion is dependent only on the agreements reached within each of the power sets.
2. If there is only one power set, then the whole group will eventually come to agree with it.
3. If there is more than one power group, the group as a whole will reach consensus only if each power group independently comes to the same consensus.