

Cellular Automata for the Network Researcher

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### Abstract

Cellular automata (CA) can be used in simulations of network processes and network evolution by identifying adjacent vertices in a network with neighboring cells in a CA. However, there are restrictions on networks that can be represented by two dimensional CAs. The degree of a vertex, the maximum number of unconnected neighbors, and the maximum size of a clique, for example, are limited. Moreover, a high degree of transitivity is built into the two dimensional CA.

Increasing the number of dimensions beyond two relaxes all the constraints. Moreover, the reduced transitivity of higher dimensional CAs allows us to vary in a systematic way the “localness” of the connections. Thus, we can examine the differences between strong (local and transitive) versus weak (non-transitive) ties, a difference that Granovetter and others have shown to be important. The paper offers equations describing how maximum degree, maximum clique size, maximum number of unconnected neighbors, and transitivity vary with the dimensions of the CA.

**KEY WORDS:** Cellular automata, networks, simulations, transitivity

## Cellular Automata for the Network Researcher

Thomas Schelling's checkerboard simulation of residential segregation is one of the best known in all the social sciences (Schelling, 1971). Schelling wanted to create a model within which he could explore the consequences of individual preferences for similar neighbors. Schelling used a checkerboard in which squares were occupied by pieces of different colors. The movements of each piece were determined by the color composition of its neighboring squares. As Schelling showed, simple rules for the actors based on their local environments could generate complex and interesting global patterns of segregation. A checkerboard is an instance of a cellular automata (CA), a grid with rules for changing the states of cells as a function of the states of their neighbors.<sup>i</sup>

CAs have also been used in simulations on the evolution of cooperation, a topic initiated by Robert Axelrod (1984). Many simulation studies have shown that the possibilities for the evolution of cooperation are enhanced when interactions are locally constrained and thus more likely to be repeated, as opposed to a random matching within a population (Axelrod, 1984, pp. 158-168).

A recent simulation study by Takahashi (2000) used the localness of CAs to model incomplete information. Takahashi first showed that if actors punished one another for their selfishness in a PD-like situation cooperation would emerge. However, as he points out, this process might work only if actors have complete information about the past behavior of all actors in the system. In a second simulation Takahashi embedded his actors in a grid network in which actors knew only the past behavior of their neighbors, rather than all actors. Such local knowledge was sufficient to produce high levels of cooperation.

Cederman (1997) simulated the development of the modern system of states within a CA. States were given opportunities to invade and incorporate other political units and to form alliances. In Cederman's simulation actors respond only to their immediate neighboring states. This reflects in part the early historical period which Cederman is trying to simulate, one in which all wars were land wars between neighbors.

The role of localness is important in Watts (1999) distinction between relational and spatial models for the small world problem. Watts explores the characteristics of networks that allow them to have *small world* properties, a combination of high clustering of choices and short paths between vertices. In spatial graphs the connection between positions is a function the distance between them in an underlying space. Watts shows that spatially-based graphs, such as lattices, cannot in general be small world graphs because local choices cannot span great distances.

The most important advantage to using cellular automata to represent networks is that they provide a natural way to represent networks that have some degree of localness; actors choose those who are near themselves or tend to cluster into groups with a high density of mutual choice. The degree of localness, or transitivity, in relationships is an important sociological variable. An actor whose contacts communicate with each other should be subject to more influence and surveillance. Laumann (1973), for example, has shown that those whose friends are friends with each other tend to hold more extreme political and social attitudes. The transitivity of ties is the basis for Granovetter's distinction between strong and weak ties. Granovetter suggests that strong intimate ties are more transitive and hence will transmit more redundant information than weak,

intransitive acquaintanceship ties, which are more likely to connect the individual to a wider variety of information sources.

This paper focuses on the effects of variations in dimension on some network parameters. It is not a complete description of all variations in cellular automata that might be of interest to network researchers. For example, it does not discuss the effects of using regular versus irregular grids (Hegselmann, Flache, and Möller 2000, Flache and Hegselmann 2001), different sizes of neighborhoods (Hegselmann 1996), synchronous versus asynchronous updating (Hegselmann 1996), discrete versus continuous outcomes (Hegselmann, Flache, and Möller 2000), or even versus odd length (Hegselmann 1996).

In this paper we will look at some of the consequences of embedding networks in lattices. Two different definitions of neighbors are examined. Neighboring cells can simply be those that are within a certain Euclidean distance of one another or, alternatively, neighboring cells can be those that are not separated from one another by intervening cells.<sup>ii</sup>

## I CELLULAR AUTOMATA WITHOUT A DISTANCE METRIC.

Let  $CA(N,k)$  be a cellular automata of  $N$  dimensions in which positions that are not separated from one another by intervening cells on any dimension and are identical on  $k$  or more dimensions are neighbors, where  $N > k \geq 0$ . The diagrams in Figure 1 show the neighborhoods of cells for different values of  $N$  and  $k$ . In the loosest definition, two neighboring cells need not share values on any single dimension ( $k = 0$ ); they need only not be separated by another cell. Using a tighter definition, neighboring cells must have equal values on at least one dimension ( $k=1$ ). By an even less flexible definition, neighboring positions must have equal values on at least two dimensions ( $k=2$ ). In three

dimensions this corresponds to sharing a corner, and edge, or a face. The cell whose neighborhood is being defined is marked with a circle and its neighbors with Xs.<sup>iii</sup>

[Insert Figure 1 about here]

A two-dimensional cellular automaton has definite limitations for the representation of networks and of network change. Depending on how a neighbor is defined, a position can have a maximum of four ( $k=1$ , *von Neumann* neighborhood) or eight ( $k=0$ , *Moore* neighborhood) connections. There is also a limit on the maximum number of partners to a position that are in turn unconnected to each other (as in a star network). The maximum number of actors unconnected to each other but connected to the same actor is four in a two dimensional CA whether the Moore or Von Neumann neighborhoods are used. This limits the centralization that can exist in a network represented by CA. A clique is defined as a set of vertices all pairs of whom are related. In a two dimensional CA no clique can have more than four members if a Moore neighborhood is used and no more than two in a Von Neumann neighborhood. Another limitation (or strength) of a two dimensional CA is its inherent localness. Cells that share neighbors will tend to be neighbors themselves. In a two dimensional CA with Moore neighbors, 3/7 of the pairs of neighbors of a cell are connected to one another.

Let's look at these constraints to see how increasing the number of dimensions alters them.<sup>iv</sup>

1. Increasing the Dimensions to Increase the Maximum Number of Neighbors

As before, let  $N$  be the number of dimensions for the CA and let  $k$  be the number of dimensions used to define a neighbor, where  $N > k \geq 0$ . The number of neighbors  $H$  of a cell is a function both of  $N$  and  $k$ .<sup>v</sup>

$$H(N, k) = \sum_{i=k}^{N-1} \binom{N}{i} 2^{N-i} \quad \text{Eq. 1}$$

The following figure shows the number of neighbors for some different combinations of  $N$  and  $k$ . Note that for as few as five dimensions a cell can have hundreds of neighbors.

[Insert Figure 2 about here]

## 2. Increasing the Dimensions to Increase the Maximum Centralization

The maximum number of neighbors of a vertex that are in turn unconnected to each other also increases with the number of dimensions. When a neighbors is defined as another positions that share only a corner ( $k = 0$ ), the maximal number of mutually unconnected neighbors of a position is  $2^N$  and when neighbors must be similar on all but one dimension ( $k = N - 1$ ), the maximum is  $2N$ . The following Table shows the maximum number of unconnected neighbors of a position for different values of  $N$  and  $k$ .<sup>vi</sup>

[Insert Table 1 about here]

Note that networks in which large numbers of unconnected vertices connect to the same point can easily be represented by higher dimensioned CA. For example, if  $N = 5$  and  $k = 1$  a position can have 18 mutually unconnected associates.

## 3. Increasing the Dimensions to Increase the Maximum Clique Size

The limitation on the maximum clique size, which is a problem when the number of dimensions  $N$  is small, ceases to be a problem as  $N$  increases. Let  $Cl(N, k)$  be the maximum clique size in an  $N$  dimensional CA when neighbors must share  $k$  dimensions.

$$Cl(N, k) = 2^{N-k} \quad \text{Eq. 2}$$

Figure 3 shows the maximal number of cliques for different values of  $N$  and  $k$ .

[Insert Figure 3 about here]

#### 4. Increasing the Dimensions to Decrease Localness

As discussed before, the transitivity associated with a CA representation of a network can actually be an advantage in the simulation of network change. However, as with the other constraints of CA, the degree of transitivity is under the control of the researcher because it decreases with  $N$ . Let  $E(N)$  be the proportion of pairs of neighbors of a cell that are in turn neighbors of each other when the neighborhood is defined in the most extensive way ( $k = 0$ ). It can be shown that:<sup>vii</sup>

$$E(N) = \frac{2 - 3^{N+1} + 7^N}{2 - 3^{N+1} + 9^N} \quad \text{Eq. 3}$$

The following graph shows the relation between the number of dimensions the tendency toward transitivity - the proportion of pairs of neighbors of a cell that are neighbors to each other.

[Insert Figure 4 about here]

Examining either the graph or the equation, it is clear that the tendency toward transitivity can be minimized to any desired level.

#### A DISTANCE-BASED DEFINITION OF NEIGHBOR

A neighbor can also be defined as any cell within a certain distance  $r$ . By this definition, a cell at the origin of a CA, with position  $(0, 0, 0, \dots)$  in  $N$  dimensions, has as neighbors all cells for which the following inequality holds:

$$\sum_{i=1}^n x_i^2 \leq r^2 \quad \text{Eq. 4}$$

To simplify the analysis I will assume a continuous space rather than a lattice so that I can calculate the volumes of spheres in spaces of various dimensions rather than the number of points or cells in a lattice.<sup>viii</sup> I will now look at the limitations and constraints.

1. Increasing the Dimensions to Increase the Maximum Number of Neighbors

There are no limitations on the number of neighbors regardless of the number of dimensions. The number of neighbors, and the volume of an  $N$  dimensional sphere with radius  $r$ , is proportional to  $r^N$ .

2. Increasing the Dimensions to Increase the Maximum Centralization

For a cell at the origin the  $2^N$  neighboring cells at the positions  $(r, 0, 0, 0, \dots)$ ,  $(-r, 0, 0, 0, \dots)$ ,  $(0, r, 0, 0, \dots)$ ,  $(0, -r, 0, 0, \dots)$ , and so forth, will not be neighbors of each other because their distances from each other are all greater than  $r$ . Thus, the number of unconnected vertices of a given vertex can be at least  $2^N$ ; by increasing the number of dimensions any star pattern can be represented.

3. Increasing the Dimensions to Increase the Maximum Clique Size.

Consider a sphere of radius  $r/2$  around a cell. All the points falling within this sphere are neighbors of the cell and are neighbors of each other. Since the volume of this sphere is proportional to  $(r/2)^N$ , there is no limitation on clique size in a CA of any dimension.

4. Increasing the Dimensions to Decrease Localness

We will first look at the two dimensional case. All the points within a circle around the origin are neighbors to a point at the origin. Select a second point within this circle and put a circle of the same radius around it. All points in the intersection of the two circles are neighbors of both points. The area of the intersection, divided by the area of either circle, is the probability that a neighbor of one is a neighbor of both. This

probability is a function of the distance between the centers of the two circles. In the following Figure the two circles have a radius of 1.00 and the distance between the two centers has been set to an illustrative .70. The area of either circle is  $\pi$ , and the area of intersection for these two circles is 1.77. The probability that a third randomly chosen neighbor of the first point is also a neighbor of the second is, therefore,  $1.77/\pi$ , or .56.

[Insert Figure 5 about here]

More generally, if the distance between the two circles is  $s$ , the two circles intersect at the points  $(s/2, (1 - s^2/4)^{1/2})$  and  $(s/2, -(1 - s^2/4)^{1/2})$ . The area of the intersection is:

$$4 \int_{\frac{s}{2}}^1 \sqrt{1-x^2} dx \quad \text{Eq. 5}$$

This divided by  $\pi$  is the probability that two neighbors at distance  $s$ ,  $1 - s \geq 0$ , will share a neighbor. The expected value of this probability will depend on the density of  $s$ . The density of  $s$  for a randomly chosen point within the circle is  $2s$ . Therefore, the expected probability that two neighbors will share a neighbor of either one is:

$$\frac{4 \int_0^1 2s \int_{\frac{s}{2}}^1 \sqrt{1-x^2} dx ds}{\pi} \quad \text{Eq. 6}$$

This expected probability is approximately .59 for the circle (in two dimensions). For  $N$  dimensions the expected probability is a generalization of this formula.

$$\frac{N \int_0^1 s^{N-1} \int_{\frac{s}{2}}^1 (\sqrt{1-x^2})^{N-1} dx ds}{\int_0^1 (\sqrt{1-x^2})^{N-1} dx} \quad \text{Eq. 7}$$

The following Figure shows these expected probabilities as a function of the number of dimensions.

[Insert Figure 6 about here]

### THREE EXAMPLES

The location of positions of power within exchange networks is a well-developed topic within social psychology (Cook et al 1983, Markovsky et al 1988). In these networks individuals must make choices among their potential exchange partners. Those in the most favorable positions have options themselves while their potential partners do not. For example, if *A* and *C* have only *B* as a potential exchange partner, *B* has power over them. On the other hand, if, in addition, *A* and *C* can transact with each other, *B*'s power disappears.

We might expect exchange networks constructed of strong ties to be more equalitarian because of their greater intimacy and positive affect. Individuals with strong intimate ties to one another should not exploit each other as much. However, this expectation ignores the structural differences between weak and strong ties. As Granovetter has pointed out (1973), strong ties are more transitive. This difference alone could affect the degree of exploitation in networks quite apart from the degree of intimacy of the ties.

Roughly speaking, tree graphs tend to produce greater power differences between positions.<sup>ix</sup> Tree graphs all contain some vertices with just one edge, and these are likely to have low power.<sup>x</sup> Tree graphs are completely intransitive. Since transitivity decreases as the dimensionality of the CA increases, simulations of exchange networks using a CA will be affected by the number of dimensions, with greater power differences being more likely with higher dimensions. Simulations using more dimensions will in effect be simulations of power within networks of weak ties. In exchange networks of strong ties

individuals will not only care more for one another; in addition there should be fewer individuals with only one possible exchange partner. This structural difference by itself should lead to less exploitation.

Bonacich (2001, 2002) conducted two kinds of simulations of the evolution of exchange networks. In these simulations actors in unprofitable positions could change their positions within the exchange networks. In a two-dimensional CA simulation, dissatisfied actors moved randomly to neighboring vacant cells. In another version, not run within a CA, dissatisfied actors randomly broke or formed ties with others. The results were strikingly different. The latter simulations, with no tendency for transitivity, took longer to achieve an equilibrium in which there were no dissatisfied actors. This was because networks without transitivity will tend to have greater power differences.

As another illustration of how the number of dimensions can make a difference, consider Axelrod's simulation of the dissemination of culture (Axelrod, 1997, pp. 148-177). One hundred agents in a two-dimensional CA were given a set of numerical traits. With probabilities that depended positively on their degree of similarity, neighboring actors were programmed to become even more similar to one another. Axelrod's main finding is that stable patterns in which different areas of the CA had different *cultures* (different values of the traits) could co-exist with one another and that the number of stable regions depended on the number of cultural features, the number of traits per feature, and the size (number of rows and columns) of the CA. I duplicated Axelrod's simulation, verifying that I got similar patterns of results in a few of the conditions he examined.<sup>xi</sup>

I suspected that the number of dimensions in the CA would affect Axelrod's findings. The greater the number of dimensions, the less the localness of the space, and the less powerful local effects should be. A two dimensional CA allows for stronger local neighborhoods capable of resisting other, perhaps larger and surrounding, neighborhoods. A CA with more dimensions should produce weaker (less transitive) and less local neighborhoods, in which people connected to the same person were less likely to be connected to each other. Thus, the ability of a neighborhood to maintain its distinctiveness should be less as the number of dimensions increases.

To examine this possibility a pair of CAs had to be selected with the same number of actors, each actor having the same number of neighbors, but with different numbers of dimensions. One of the CAs I chose had two dimensions with twelve rows and columns, or 144 cells. The Moore neighborhood was used, so that each actor had eight neighbors. The other CA had four dimensions. In this CA neighboring cells had to share three dimension ( $CA(4,3)$ , using the previous symbols). In this CA each cell also has eight neighbors. The number of separate values for the four dimensions was 4, 4, 3, and 3, which also generates 144 cells.<sup>xii</sup>

The localness of the two CAs is quite different. Using equation 3, 3/7 of the neighbors in the two-dimensional CA are neighbors of each other, but none of the neighbors of a cell in the four-dimensional CA are neighbors of each other. In Granovetter's (1983) sense, the ties in the four-dimensional CA are weaker.

The differences between simulations in the two CAs were striking. Ten simulations were run in the two-dimensional and four-dimensional CAs. In the two-dimensional CA there were, on average, 25.5 stable cultures, while in the four-

dimensional CA the average number was 3.0. The average size of the largest culture was 73.5 out of 144, slightly more than half, in the two-dimensional CA, but 131.1 out of 144 in the four-dimensional CA. In sum, local neighborhoods in the two-dimensional CA were easier to establish because the forces of locality were stronger.

As a third example, Michael Chwe (2000) used the dimensionality of a cellular automata to show the effects of strong versus weak ties on simulated spread of social movements. In Chwe's model actors must be assured of common knowledge about each other's intentions before they act. Chwe identifies low and high dimensionality with strong and weak ties respectively because of the differences in transitivity. Chwe shows that small *cliques* (completely connected sets) are more easily achieved in low dimensional CAs while large cliques are more easily achieved in higher dimensions. Thus, the degree to which a social movement spreads successfully through strong or weak ties will be a function of how much reassurance people need about the intentions of others before they act.

## CONCLUSIONS

Networks can be represented in cellular automata by identifying vertices adjacent in the graph with neighboring cells in the CA. Limitations in the networks that can be represented are reduced by increasing the number of dimensions in the CA to more than two. As the number of dimensions increases the maximum degree for any vertex, the maximum number of unconnected neighbors, and the maximum clique size increase. In fact, any graph can be represented within a CA with sufficient dimensions.

The more important point discussed in this paper is that the degree of localness (transitivity) can be manipulated by varying the number of dimensions in a simulation.

Granovetter's distinction between strong and weak ties is based on differences in transitivity; strong intimate ties, based on similarity and involving frequent contact, are more transitive. Embedding a simulation in CAs of varying dimensions is a way of examining the effect of strong versus weak ties in a network process.

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Table 1. Maximum Number of Unconnected Neighbors for Dimensions  $N$  and Neighbor

Definition  $k$

k	1	2	3	4	5	6	7
0	2	4	8	16	32	64	128
1		4	6	10	18	34	66
2			6	8	16	24	40
3				8	10	32	32
4					10	12	56
5						12	14
6							14

FIGURE 1. Neighborhoods for different values of  $N$  and  $k$ .

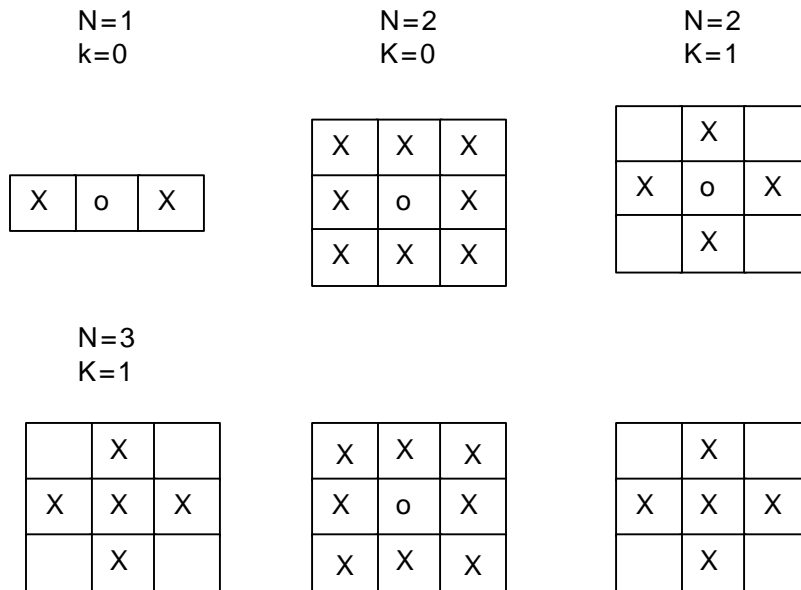


FIGURE 2. Number of neighbors in  $CA(N,k)$

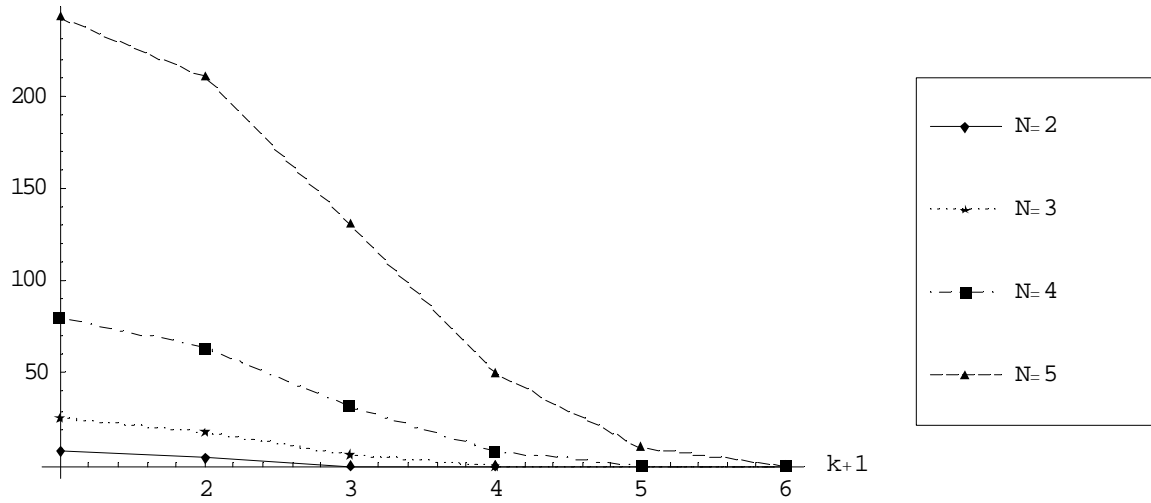


FIGURE 3. Maximum clique size in  $CA(N,k)$

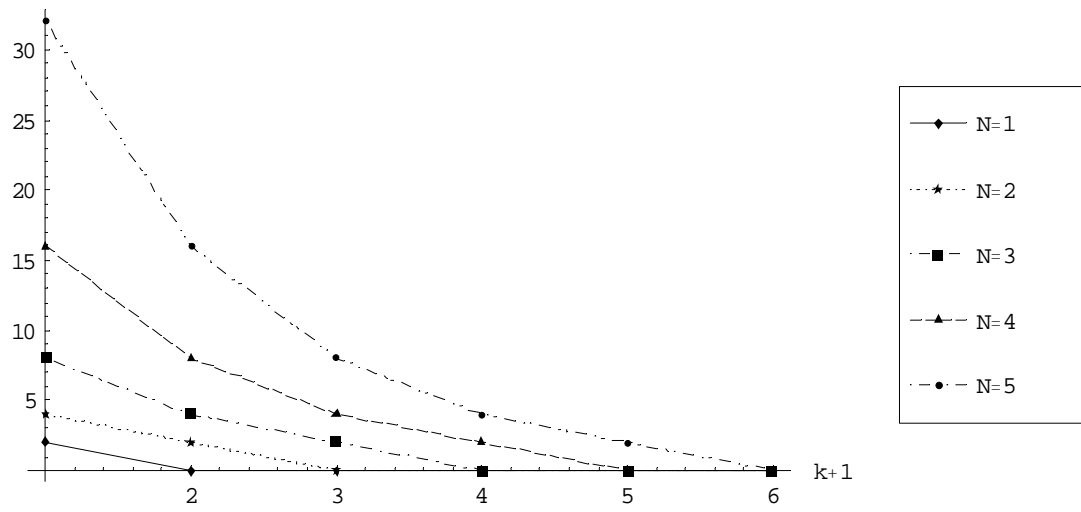


FIGURE 4. Proportion of neighbors unconnected to each other in  $CA(N,0)$

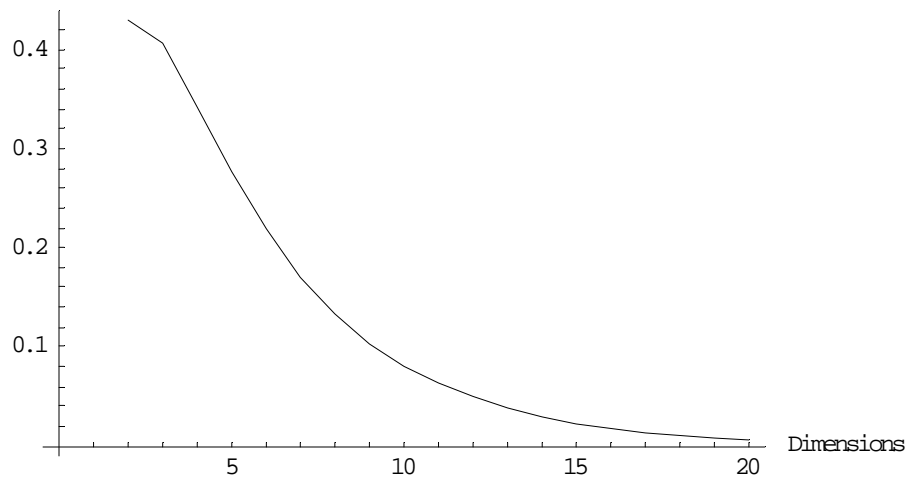


FIGURE 5. Intersection of Two Unit Circles Separated by .70

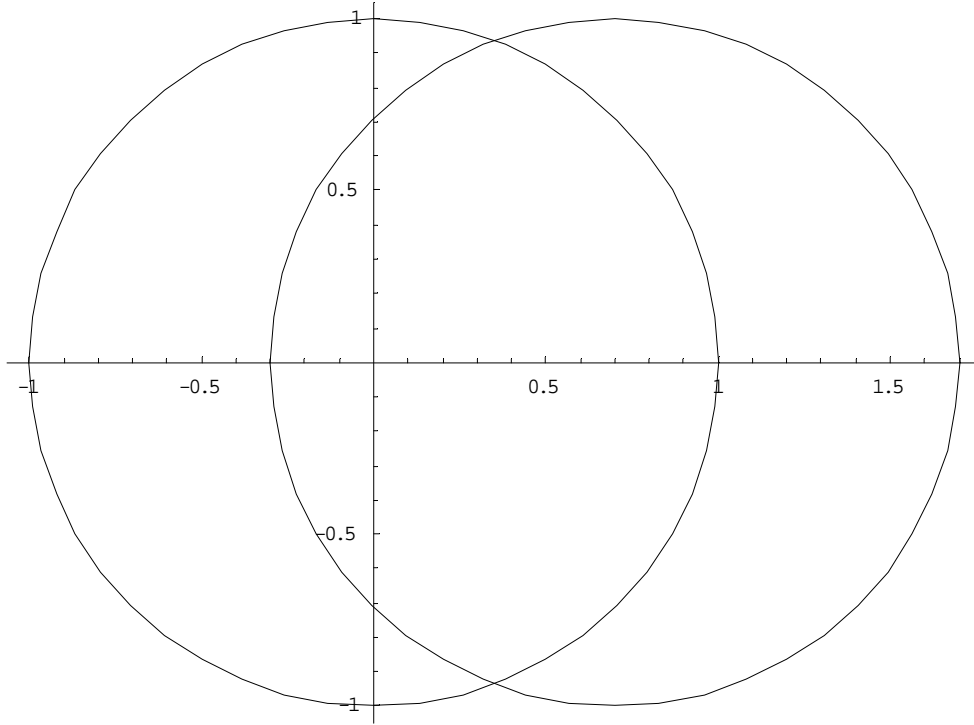
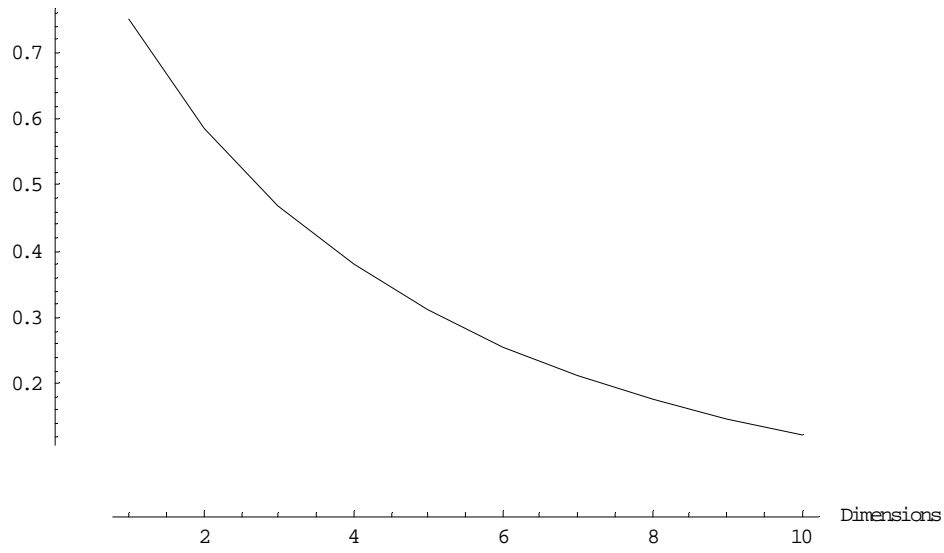


FIGURE 6. Proportion unconnected neighbors in distance-based neighborhoods related to dimensions.



## Appendix: Proofs of Equations

### EQUATION 1

Neighbors are alike on at least  $k$  of the  $N$  dimensions. The binomial coefficient  $N!/i!(N-i)! = {}_N C_i$  gives the number of ways in which  $i \geq k$  dimensions of similarity can be chosen from the  $N$  dimensions. The neighbors can be immediately above or below the given cell on the remaining  $N - i$  dimensions; hence the  $2^{N-i}$ .

### EQUATION 2

If a set of cells are identical on  $k$  dimensions they can differ only on the remaining  $N-k$ . Since they cannot be separated on any of these dimensions, there are only two neighboring values that are possible on each of them.

### EQUATION 3

Equation 3 is a simplification of the following formula:

$$\frac{\sum_{k=1}^N \binom{N}{k} \frac{\left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{N-k}}{1 - \left(\frac{1}{3}\right)^N} (2^k 3^{N-k} - 2)}{3^N - 2}$$

This formula is based on the binomial distribution. The formula refers to the neighborhood of a single orienting cell. Every cell in this neighborhood may share values on some dimensions with the orienting cell and have different values on other dimensions.  $k$  is the number of dimensions in which a cell has a different value than the orienting cell. The  $k$  dimensions may be selected from the  $N$  dimensions in  ${}_N C_k$  different ways. The probability that any other neighboring cell, the focal cell, differs from the

orienting cell in these  $k$   $N$ - dimensions is  $(2/3) (1/3)^{N-k}$ , since every cell in the neighborhood has a value either equal to, one greater

The denominator  $1 - 2^{-N}$

be the center of an  $N$

neighborhood with  $2^k - 2$  cells, 2 subtracted for the focal cell and for the orienting cell.

$2^k - 2$ -orienting cells. The

$N$  neighbors of the

of the focal cell. Surprisingly, this equation

simplifies to equation 3.

#### EQUATION 5

The integral is the area under the curve between the values 1 and  $s$  one quarter of the area of intersection.

6

The density function for the distance from the origin of a randomly chosen point  $r$ . Larger values of  $r$  are more likely because the area of a circle increases at a rate proportional to its radius

#### EQUATION

$r^N$  is proportional to the density function for the distance of a randomly selected point  $r$  is proportional to the overlap in volume between the random circle and the initial circle.

The denominator is proportional to the volume of  $N$  dimensions.

## NOTES

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<sup>i</sup> I will use CA as a singular word even though "cellular automata" is plural in form.

<sup>ii</sup> Neighborhoods can also be defined in a more continuous manner. Hegselmann (1998)

ein (1966) , for example, allow the influence of positions on one

Defined in this way, neighborhood does not correspond to any Minkowski distance

Minkowski metrics are a family of distance measure

Euclidean distance measure  $d = \sum |x_i - y_i|^r$

when  $r = 2$ . For example, consider two cells  $A$  and  $B$  that are identical on every

side two. By any Minkowski

metric above because there is a cell separating them. Now consider another position that is not

adjacent to  $A$ . It is a neighbor (if  $r = 0$ ), but its

distance is  $N$ , where  $N$  is a Minkowski coefficient. For some values of  $N$  and  $r$ ,

<sup>iv</sup> Every graph can be represented by a cellular automata if the number of dimensions is

at least  $n$  whose columns correspond to the vertices of the graph

*not*

the number of such unconnected pairs. In each row enter a "+1" and a "-1" in the

columns of unconnected vertices of that row. Enter a zero in all the

$N$

columns of a  $N$ -dimensional CA. Use the definition of neighbor corresponding to  $Ca(r,0)$ . All

unconnected pairs fail to be neighbors in the CA, while all connected

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<sup>v</sup> Proofs for the equations are in an appendix.

<sup>vi</sup> I was not able to find a general formula.

<sup>vii</sup> I do not have a formula for different values of  $k$ .

<sup>viii</sup> For example, in a 2-dimensional lattice any sphere with a radius greater than or equal to the square root of two and less than two will include all the Moore neighbors.

<sup>ix</sup> A tree graph is connected and has no circuits (Busacker and Saaty 1965).

<sup>x</sup> There are trees without great power differences and non-trees with great power differences, but, on the whole, there is an association between the two. For instance, the only connected graphs with three and four vertices that have great power differences between vertices are trees.

<sup>xi</sup> The only difference was that Axelrod's simulation was on a checkerboard with a boundary, so that positions on the edges or corners had three and two neighbors respectively. My simulations were on a torus, so that the top and bottom and the two sides were joined and there were no edges or corners. I tried to duplicate two of the analyses reported in Table 7.2 (page 160): the simulations with five cultural features and 10 and 15 traits per feature. I found 1.8 (instead of 3.1) average number of stable regions with 10 traits and 5.7 instead of 20.0 regions with 15 traits. The pattern is the same (more regions with more traits). The differences in absolute numbers may have been due to the fact that Axelrod's CA had borders. The average distance between cells in a checkerboard with borders is greater than in a torus with the same number of cells.

<sup>xii</sup> I used four cultural traits and 21 traits per feature.