Depth versus Rigidity:
R Simulation to Demonstrate Equilibrium Existence
January 16, 2012

Suppose that $u(t) = \ln(t)$ and $\beta = 1$. Then $t_D(a) = a$, $t_S(a) = a - \sigma$, $a_B = t_B$, and $a_S = t_B + \sigma$. So the proposed equilibrium behavior is:

$$t_E(a) = \begin{cases} a & \text{if } a < t_B \\ t_B & \text{if } t_B \leq a \leq a_S \\ a - \sigma & \text{if } a_S < a < a_D \\ a & \text{if } a_D \leq a \end{cases}$$

Then:

$$\begin{aligned} \bar{\Delta}(a) &= EU\left(S|t_S(a), a\right) - EU\left(D|t_D(a), a\right) \\ &= a\left[\ln\left(a - \sigma\right) - \ln\left(a\right)\right] - \sigma\left[\ln\left(a - \sigma\right) - \ln\left(t_B\right) - 1\right] + \delta H\left(a_D\right)\left(\chi_C - \chi_N\right) \end{aligned}$$

Note that:

$$\chi_N = \frac{1}{1-\delta} \int_0^A \left[(a-1)\ln(a) - a\right] dH(a)$$

and:

$$\begin{aligned} \chi_C &= \int_0^A \left[au\left(t_E(a)\right) - t_E(a)\right] dH(a) - \sigma \int_{a_S}^{a_D} L^*\left(t_E(a)\right) dH(a) - \int_0^A u\left(\tau_E(\alpha)\right) dH(\alpha) \\ &\quad + \sigma \int_{\alpha_S}^{\alpha_D} L^*\left(\tau_E(\alpha)\right) dH(\alpha) + \delta H\left(a_D\right)^2 \chi_C + \delta\left[1 - H\left(a_D\right)^2\right] \chi_N \\ &= \frac{1}{1 - \delta H\left(a_D\right)^2} \left\{ \int_0^A \left[(a-1)\ln\left(t_E(a)\right) - t_E(a)\right] dH(a) + \delta\left[1 - H\left(a_D\right)^2\right] \chi_N \right\} \end{aligned}$$

$$\begin{aligned} \Rightarrow \chi_C - \chi_N &= \frac{1}{1 - \delta H\left(a_D\right)^2} \left\{ \int_0^A \left[(a-1)\ln\left(t_E(a)\right) - t_E(a)\right] dH(a) - (1-\delta)\chi_N \right\} \\ &= \frac{\Psi\left(a_D\right)}{1 - \delta H\left(a_D\right)^2} \\ \text{where } \Psi\left(a_D\right) &= \int_{t_B}^{a_S} \left[(a-1)\left[\ln\left(t_B\right) - \ln(a)\right] + a - t_B\right] dH(a) \\ &\quad + \int_{a_S}^{a_D} \left[(a-1)\left[\ln\left(a - \sigma\right) - \ln(a)\right] + \sigma\right] dH(a) \end{aligned}$$

So the equilibrium exists if there is a value of $x \in (a_S,\, A)$ that solves:

$$\lambda(x) \quad = \quad x\,[\ln(x-\sigma)-\ln(x)]-\sigma\,[\ln(x-\sigma)-\ln(t_B)-1]+\frac{\delta H(x)\,\Psi(x)}{1-\delta H(x)^2}=0$$

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Recall that $a \sim U[0, A]$. This implies that $h(x) = \frac{1}{A}$ and $H(x) = \frac{x}{A}$. For the purposes of computer simulation, assume that $A = 5$, $t_B = 3$, $\sigma = 0.75$, and $\delta = 0.9$. Then the following R code demonstrates that $x \approx 4.96$ solves $\lambda(x) = 0$:

```
A ← 5
tB ← 3
sigma ← 0.75
delta ← 0.9
aS ← tB + sigma

y ← seq(tB, aS, length = 10000)
integrand1 ← ((y − 1) ∗ ((log(tB) − log(y)) + y − tB) ∗ (1/A)
integral1 ← sum(integrand1)/length(integrand1)

x ← seq(aS, A, length = 10000)
integral2 ← c(rep(NA, length = length(x)))

for(i in 1 : length(x)) {
  z ← seq(aS, x[i], length = 10000)
  integrand2 ← ((z − 1) ∗ (log(z − sigma) − log(z)) + sigma) ∗ (1/A)
  integral2[i] ← sum(integrand2)/length(integrand2)}   }

psi ← integral1 + integral2
lambda ← x ∗ (log(x − sigma) − log(x)) − sigma ∗ (log(x − sigma) − log(tB) − 1)
  + (delta ∗ (x/A) ∗ psi)/(1 − delta ∗ ((x/A)^2))

summary(lambda)
lambda[9682]
lambda[9683]
x[9682]
x[9683]
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