Homework 8 Answers   PS 30   December 2013

1. Say that we have a threshold model in which there are 5 people. If the total number of other people who participate is greater or equal to a person's threshold, the person wants to participate also. If the total number of other people who are participating is less than a person's threshold, the person does not want to participate.

a. Say that one person has threshold 1, two people have threshold 2, and two people have threshold 4. Find all of the pure strategy Nash equilibria.
One Nash equilibrium is (n, n, n, n, n), in which no one participates—since no one participates, no one's threshold is met. Another is (p, p, p, p, p), in which everyone participates—everyone's threshold is met (for example, the threshold 4 people see 4 other people participating). Another is (p, p, p, n, n)—the threshold 1 and the two threshold 2 people participate because their thresholds are met.

b. Now say that one of the threshold 2 people becomes a threshold 0 person. Find all of the pure strategy Nash equilibria. Does this change guarantee some level of participation?
The threshold 0 person participates for sure. Hence the threshold 1 joins in, and then the threshold 2 person joins in. However, a threshold 4 person does not join in because there are only three other people participating. So one Nash equilibrium is (p, p, p, n, n). Another is (p, p, p, p, p)—if everyone starts off by participating, then everyone wants to continue to participate. The outcome (n, n, n, n, n), however, is no longer a Nash equilibrium. Hence the change guarantees at least some level of participation.

2. Say that you have a group of 50 people who can either buy a color fax machine or not buy. No one wants to buy a color fax machine if no one else has one (because there would be no one to exchange color faxes with). In fact, each person will buy one only if at least 6 other people buy them. Thus each person has a threshold of 6.

a. Find the two pure strategy Nash equilibria.
The pure strategy Nash equilibria are $(p, \ldots, p)$ (everyone participating) and $(n, \ldots, n)$ (no one participating).

b. Now say that you are a sales rep for the color fax machine company. You can offer discount coupons to potential customers. If you give 1 discount coupon to someone, that decreases their threshold by 1. For example, if you give 6 discount coupons to a single person, you can make that person have threshold 0. If you give 2 discount coupons to a single person, you can make that person have threshold 4. Obviously, you can guarantee that everyone will buy a color fax machine by giving all 50 people six coupons each, but that would be silly (the company would get no profits). Using the fewest possible number of coupons, how can you guarantee that everyone will buy a color fax machine?
To guarantee that someone will participate for sure, you need to have one person who has threshold 0 (you give that person 6 coupons). Given that this person participates, you need to make one person have threshold 1 (you give that person 5 coupons). Similarly, you give 4 coupons to another person to make that person have threshold 2, you give 3 coupons to make another person have threshold 3, and so forth. Hence you give out 6 coupons to 1 person, 5 coupons to another person, 4, to another person, 3 to another person, 2 to another person,

and 1 to another person. So the resulting thresholds are $0, 1, 2, 3, 4, 5, 6, 6, 6, \ldots, 6$. The first six people (with thresholds 0 to 5) will revolt, and then everyone else's threshold will be met and then everyone else will revolt. You give out a total of $6 + 5 + 4 + 3 + 2 + 1 = 21$ coupons.

3. [from Spring 2003 final] Say that we have 10 people. Each person is thinking about whether or not to join a revolt or not. Each person has a threshold: five people have threshold 4 and five people have threshold 6.

a. Find all pure strategy Nash equilibria of this game.
Everyone not revolting is a Nash equilibrium; if no one revolts, no one wants to revolt. If someone revolts, at least five people must revolt (since the lowest threshold is threshold 4). If at least five people revolt, then all of the threshold 4 people revolt. If all five threshold 4 people revolt, this alone is not enough to get the threshold 6 people to revolt. So another Nash equilibrium is the threshold 4 people revolting and the threshold 6 people not revolting. Say one of the threshold 6 people revolts; then all of the threshold 6 people want to revolt. Hence another Nash equilibrium is everyone revolting.
So there are three Nash equilibria: no one revolts, only the threshold 4 people revolt, and everyone revolts.

b. Say that you have some discount coupons which lower the cost of revolting and hence lower a person's threshold. For example, if I give 3 coupons to a person with threshold 4, she now has threshold 1. If I give 1 coupon to a person with threshold 6, he now has threshold 5. By giving out coupons, I can change the game so that the only Nash equilibrium is one in which everyone revolts. I want to do this by giving out the fewest number of coupons. How do I distribute the coupons (who gets coupons, and how many does each person get)?
To get rid of the Nash equilibrium in which no one revolts, someone must have threshold zero. The "cheapest" way to make someone have threshold zero is to give 4 coupons to a threshold 4 person. To make another person revolt for sure, we need to make her have threshold 1; so we can give 3 coupons to a threshold 4 person. Similarly, we make another threshold 4 person have threshold 2 by giving him 2 coupons and make another threshold 4 person have threshold 3 by giving him 1 coupon. Finally, we give a threshold 6 person 1 coupon and make him a threshold 5 person. So in other words, we started with thresholds $4, 4, 4, 4, 4, 6, 6, 6, 6, 6$ and we end up with thresholds $0, 1, 2, 3, 4, 5, 6, 6, 6, 6$. We give out $4 + 3 + 2 + 1 + 0 + 1 = 11$ coupons.

c. Now say that you can give tickets which raise the cost of revolting and hence raise a person's threshold. For example, if I give 3 tickets to a person with threshold 4, she now has threshold 7. If I give 2 tickets to a person with threshold 6, he now has threshold 8. By giving out tickets, I can change the game so that the only Nash equilibrium is one in which no one revolts. I want to do this by giving out the fewest number of tickets. How do I distribute the tickets (who gets tickets, and how many does each person get)?
To eliminate the equilibrium in which everyone revolts, there must be at least one person with threshold 10. Given that this person never revolts, you can make another person never revolt by making them have threshold 9, and so forth. So we start with thresholds $4, 4, 4, 4, 4, 6, 6, 6, 6, 6$ and end up with thresholds $4, 4, 4, 4, 5, 6, 7, 8, 9, 10$. We give out $1 + 0 + 1 + 2 + 3 + 4 = 11$ coupons.

4. Say that there are four people: Alicia, Betsy, Carlos, and Davis. Each can choose whether to wear platform sandals or not. Alicia is very fashion-forward and will wear them even if no one else wears them; in fact, if more than one other person wears them, she won't wear them anymore because she hates being part of a crowd. Betsy also likes fashion but is not as cutting-edge: she will wear them if at least one other person wears them, but like Alicia, hates being "part of the crowd" and will not wear them if more than two other people wear them. Carlos thinks of himself as hip, but is kind of slow on the uptake and will wear them if at least two others wear them. Still, Carlos has at least some fashion pride and will not wear them if everyone else wears them. Finally, Davis gets his fashion tips from the JC Penney catalog and will wear them if everyone else wears them.

a. Say that at the beginning, no one wears platform sandals (they have just hit the market). Show that at first the sales of platform sandals steadily grow, but eventually sales "cycle" between high and low in a never-ending "fashion cycle."
Starting from (n, n, n, n) (no one buys), first Alicia only wears platform sandals. Hence we get (p, n, n, n). Given this, Betsy jumps in and we get (p, p, n, n). Given this, Carlos jumps in and we get (p, p, p, n). Now Davis jumps in but Alicia drops out because they are becoming too popular; hence we get (n, p, p, p). Now Davis drops out and we get (n, p, p, n). Now Carlos drops out and we get (n, p, n, n). Now Alicia jumps back in and Betsy drops out, and we get (p, n, n, n). Then the cycle starts all over again.

b. Are there any pure strategy Nash equilibria of this game?
There are sixteen possibilities to consider (p, p, p, p), (p, n, n, n), and so forth. To avoid having to go through all sixteen, we first realize that the only way that Davis participates is if everyone else participates. It is easy to show that (p, p, p, p) is not a Nash equilibrium (Alicia wants to drop out). Hence we can assume that Davis never participates. Given this, Carlos participates only if Alicia and Betsy participate: (p, p, p, n). It is easy to see that this is not a Nash equilibrium (Alicia wants to drop out). Hence we can assume that Carlos does not participate. So there are only four possibilities: (p, p, n, n), (p, n, n, n), (n, p, n, n), and (n, n, n, n). It is easy to check that none of these is a Nash equilibrium. So there are no pure strategy Nash equilibria of this game.

5. Say that there are three men A, B, and C and three women X, Y, and Z. Each of these six people is considering matching up with a member of the opposite sex. Each person would rather be matched with someone than not have a partner at all. Man A prefers woman X best, woman Y next, and woman Z least. Man B prefers woman Y best, woman Z next, and woman X least. Man C prefers woman Y best, woman X next, and woman Z least. Woman X prefers man B best, man A next, and man C least. Woman Y prefers man A best, man B next, and man C least. Woman Z prefers man A best, man C next, and man B least.

a. Say that man A is matched with woman X, man B is matched with woman Y, and man C is matched with woman Z. Is this matching stable?
Yes, this match is stable. Men A and B get their first choice. Man C would like to call X or Y, but both X and Y like their current partners over C. Woman X would like to call B, but B is attached to his first choice. Women Y and Z would like to call A, but A is attached to his first choice.

b. Write down all possible matchings and determine which of them are stable and which are not stable.

(AX, BY, CZ) is stable, as mentioned above. (AX, BZ, CY) is not stable—B and Y would like to dump their current partners and join with each other. (AY, BX, CZ) is stable. (AY, BZ, CX) is not stable—A and X would like to get together. (AZ, BX, CY) is not stable—A and Y would like to get together. (AZ, BY, CX) is not stable—A and Y would like to get together.

c. Among the set of stable matchings, which matching is most preferred by the men? Among the set of stable matchings, which matching is most preferred by the women?

All of the men prefer the stable matching (AX, BY, CZ) at least as much as the other stable matching, (AY, BX, CZ). All of the women prefer the stable matching (AY, BX, CZ) at least as much as the other stable matching, (AX, BY, CZ).

6. Say that there are four men, A, B, C, and D and four women W, X, Y, and Z. Each of these eight people is considering matching up with a member of the opposite sex. Each person would rather be matched with someone than not have a partner at all. Man A prefers woman W best, X next, Y, next, and Z least. Man B's preference ordering (from best to worst) is Y, X, W, Z. Man C's preference ordering is Y, Z, X, W. Man D's ordering is Y, Z, X, W. Woman W's preference ordering (from best to worst) is D, C, B, A. Woman X's preference ordering is C, B, A, D. Woman Y's ordering is D, A, C, B. Woman Z's ordering is B, C, D, A.

a. Among the set of stable matchings, which matching is most preferred by the men? Among the set of stable matchings, which matching is most preferred by the women?

To find the stable matching most preferred by the men, we use the "men ask" algorithm. First everyone asks their first choice: A asks W, B asks Y, C asks Y, and D asks Y. Woman Y likes D best out of all the men who ask her, and thus B and C are rejected. So B and C ask their second choices: B asks X and C asks Z. As it stands, A is matched with W, B with X, C with Z, and D with Y, and there are no women who are asked by more than one man. Hence (AW, BX, CZ, DY) is a stable matching.

Now let's use the "women ask" algorithm. First W asks D, X asks C, Y asks D, and Z asks B. Man D is asked by both W and Y, and so D rejects W and pairs up with Y. So W then asks C. Now C is asked by two people (W and X), and C chooses to pair up with X. So the current matches are XC, YD, and ZB.

Since W is rejected again, W asks B. Now B is asked by W and Z, and B chooses to pair up with W. So the current matches are now WB, XC, and YD. Since Z is rejected, she asks C. Now C is asked by X and Z, and C chooses Z. Now the current matches are WB, YD, and ZC. Since X is rejected, she asks B. Now B is asked by both W and X, and B chooses X. Now the current matches are XB, YD, and ZC. Now W asks A. Since no man is now asked by more than one woman, the matching (WA, XB, YD, ZC) is stable.

Note that this matching is the same as what we obtained in the "men ask" algorithm. Since the best matching for the women is the same as the best matching for the men, in this case this matching is the only stable matching.