Bargaining with a Claims Structure: Possible Solutions to a Talmudic Division Problem

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July 30, 2009

Abstract

Abstract: A man wants to sell his land, but his two wives hold claims to acquire it in case of a divorce or his death. The first wife writes the buyer giving up her claim and the sale goes through. After the husband dies, the second wife points out that she did not relinquish her claim and she seizes the land from the buyer. Then the first wife takes the land from the second wife, arguing that she has priority over the second and gave up her right only with respect to the buyer. The buyer then seizes the land from the first wife, and so on. The Talmud states that they compromise but does not give their shares. The situation can be construed as a bargaining game with non-transferable utility, assuming that side payments can be seized just like the land, or as a sequential-offer bargaining game. For a reasonable characteristic function the NTU-value of Harsanyi seems most intuitive.

Acknowledgement: The author thanks Renee Bowen, Dov Samet, Catherine Ross, David Schmeidler and Muhamet Yildiz for helpful discussions, and is grateful for support from the Leon Levy Foundation and the Institute for Advanced Study in Princeton.

1. The problem

The Talmud, the compendium of Jewish laws written down during the first few centuries of the common era, includes various rules for dividing a commodity among claimants. Some of these and others in the later Talmudic literature have been studied using game-theoretical methods (e.g., Aumann and Maschler 1985, on three wives holding marriage contracts; O'Neill 1982, on four sons sharing an inheritance.) The research has contributed to a body of work on fair division in the face of conflicting claims (Thomson 2003) and some of its results have been presented in the community of Talmudic scholars (Aumann 1999).

Each rule in the Talmud is structured as a *mishnah*, or teaching, surrounded by the *gemara*, a debate and explanation by the rabbis. The book on marriage contracts (*ketubot*) includes the following *mishnah*. (The bold type gives the wording of the original Aramaic and the rest is the editors' elaboration.)

> If a man had two wives, and the *ketubah* deed of one was dated earlier than that of the other, **and** the husband **sold his field** over which both [of] them exercised a lien, but which was not sufficient for both *ketubah* settlements, **and the** man's **first** wife, who had the preferential lien, **wrote to the buyer** of the property, "**I will have no claim against you** regarding this field," thereby renouncing her right to seize that field from him as payment of her *ketubah* settlement, and the husband later died, **the** man's **second** wife **can take** the field **away from the buyer**, for she never waived her right to collect her *ketubah* settlement from that property. **And** then **the** man's **first** wife can remove the field **from the second** wife for she has the preferential right of recovery and did not waive her right to collect from the second wife. **And** then **the buyer** can remove the field **from the first** wife, for she had waived her right to dispute with him over that field, **and the cycle repeats itself** over and over again the second wife seizes from the buyer, the first wife from the second wife, and the buyer seizes from the first wife **until the three** parties make a **compromise among themselves** and come to an agreement about how to divide the field . . . . (*Babylonian Talmud*, Steinsaltz edition, *Ketubot*, 95a.)

In other division problems the problem is that the parties are claiming different amounts that sum to more than is available, but here each party has a claim on the whole property and directs it only at certain other parties. The claim becomes active only when that person holds the property. Also, the final division is achieved by the parties bargaining rather than an outsider choosing a division. In another famous passage (*Baba Mezi'a* 2a), two disputants argue over a garment found on the street and it is the court that settles it, but here they solve it themselves.

In other division problems the parties are claiming different amounts which sum to more than the total available and the question is the proper size of their shares. In this case each party has a claim on the whole property but one directed only at certain other parties. The claim becomes active when its target holds the property. Also, the solution is achieved by bargaining rather than by an outsider choosing a division. In another famous passage (*Baba Mezi'a* 2a), two disputants argue over a garment found on the street and it is the court that settles it, but here they are told to solve it themselves.

The dilemma arises here because the buyer got a release from only one wife. Why did he not insist on letters from the second? The Talmud's situations are sometimes far-fetched. For example, the passage analyzed by Aumann and Maschler (*Ketubah* 93a) had the man marry his three wives on the same day and by messenger. The standard resolution for conflicting marriage claims was that the first wife took everything up to her share, then whatever was left went to the second up to her share, then to the third, etc., but the Talmud's setup ensured that no one knew who was first.) The present problem could be one of these, more aimed at clarifying the logic than giving a practical rule. The broad structure of cyclical claims had other applications since, as the Talmud and later commentators note, cycles can arise from mechanisms involving several creditors or in-laws (Naeh and Segal 2008.)

On the other hand, perhaps he doubted that the second wife would cooperate and even if he had succeeded the price might have risen, so he took a chance that she would not need the land to satisfy her marriage contract. The *gemara*, the surrounding discussion, is unhelpful on choosing a specific division. The rabbis worry about whether the first wife wrote her letter just to please her husband and whether it has any legal effect. Perhaps they thought the exact shares were obvious: the claims are in a circle so each party should get one-third. However the situation is not really symmetrical since the buyer holds the land and can enjoy it until the court's verdict; he could ask for a bigger share in exchange for an immediate deal. Even if an equal division were obvious for this problem, adding another wife would make the claims pattern asymmetrical, as Figure 1(c) shows. Now wife 2 is in a stronger position than wife 3 since both she has the same incoming claim as 3, but her outgoing claims are a proper superset of 3's. Equal division seems inappropriate, so we need a general theory.

An *n-person claims structure*, $C = (N, E, T, h)$, is a set $N = \{1, ..., n\}$ of parties, a set $E \in 2^{N \times N}$ of directed edges, a positive real number T interpreted as the total available, and a holder $h \in N$. In Figure 1 the parties are represented as points, the holder is circled, and an edge, or arrow, means that the tail party has a claim to take the land from the tip party. The ordered pair representing an edge is written $i \to j$. Conditions on E are $i \to i \notin E$ (no arrow starts and ends at the same node), and not both $i \to j \in E$ and $j \to i \in E$ (claiming cannot be mutual.) For $j \in N$ the members of the set $\{i \in N | i \to j\}$ are the *claimants* on j. A pattern (N, E) without a distinguished holder is called a *graph*, and, following graph theory terminology, if either $i \to j \in E$ or $j \to i \in E$ for all distinct i, j, the graph is a *tournament.* A *solution* Ψ is function assigning to each C of a set of n-vectors, each satisfying $x_i \geq 0$ and $\sum x_i = 1$. These are interpreted as possible vectors of reasonable payoffs. When the set of solutions is single-valued solution" will be used to refer to the payoff vector itself.

FIGURE 1 HERE

Figure 1 shows some claims problems for three and four persons (two or three wives plus a buyer.) The three-person line, Figure 1(a), is simple but not at all trivial, and will provide a test case for various solutions. Figure 1(b) is the Talmud's three-person cycle, and Figure 1(c) adds a third wife, who keeps her claim as the second does.

One could generalize the situation by starting with the land distributed among several people or by allowing unequal claims. Figure 2 shows how the latter case might encompass Aumann and Maschler's problem, *Ketubah* 93a. This analysis, however, will treat one unit of land, initially in the hands of one person, with parties claiming the entire amount.

FIGURE 2 HERE

Section 2 raises two obvious solutions but argues against them. Section 3 treats the problem as an NTU bargaining game by defining a characteristic function and calculating solutions with the values of Harsanyi (1963), Shapley (1969), and Maschler and Owen (1989, 1992), as well as with an NTU version of the nucleolus. Section 4 defines a sequential bargaining game. Section 5 compares the solutions according to whether payoffs respond positively to certain changes in the claims problem. The conclusion uses the results to comment on the coalitional and strategic approaches to bargaining.

2. Two unacceptable solutions

The Talmud suggested that the land might get passed around forever, and this notion could be developed as follows. In each period we identify the parties with claims on the current holder and we transfer the land to each of them equiprobably and independently of previous transfers. The *Markov solution* is the vector x^* of steady-state likelihoods that each party is holding it, or, equivalently, the long-term proportions of the time each holds it. For graphs like those in Figure 1 where these proportions are independent of the initial holder, the solution is determined by $x^* = x^*\mathbf{M}$, where $\mathbf{M}$ is the transition matrix. The transition matrices and solutions for Figure 1's problems are shown in Figure 3.

FIGURE 3 HERE

The Markov solution has the desirable feature that it removes players' incentives to renegotiate. After the land has been distributed, the parties could be seen as facing a set of new problems, one over each portion of the land in someone's possession, that portion being subject to the claims of the original graph. If we solve these by the Markov method, their total will have their original allocation. This property is not unique to the Markov solution but holds for any solution that covaries in ratio with the amount available and that chooses an allocation that is independent of the original holder whenever the latter is among those who get a positive share.

The fixed point property is attractive, but the Markov solution is unacceptable for strategic bargainers. In the three-person line the land would go from Buyer 3, to Wife 2, then to Wife 1 and stay there, for a division $(1, 0, 0)$. In fact Wife 2 is in a strong position: she can refuse to seize the land from 3 unless 1 allows her some positive share. If the alternative were the Markov solution, her carrying out this threat would be costless to 2 but would block any gain by 1, so the latter should offer her something for her cooperation. However the Markov solution has 2 seizing the land without considering what will happen next. As to the rationale of the fixed point property, one can argue against it since the output of a solution rule is an agreement that the parties have bound themselves to stick with, and should not be the input for further claiming.

Another proposal would be to calculate the straightforward Shapley TU-value. In case of no agreement two opposing coalitions would form and if the current holder were subject to a claim from someone in the opposing side, then that player would transfer the land elsewhere in his own coalition in exchange for a side payment. In the three-person cycle with the coalition partition 13 versus 2, player 3 could keep the land away from 2 by giving it to 1 in exchange for a payment, so the characteristic function would have values $v(13) = 1$ and $v(2) = 0$. Also $v(12) = 1$ since Wife 2 can grab the land from 3 and hold it as long as she pays her coalition partner 1 not to take it from her. Similarly $v(23) = 1$. The characteristic function for the Talmud's three-person cycle assigns 0 to single players, 1 to pairs, and 1 to the grand coalition, so by symmetry the Shapley value gives an equal division. For the three-person line of 1(a) the division would also be equal, and for the 4-person tournament of 1(c) it would be $(5, 3, 1, 3)/12$.

An objection is that in a typical real legal system side payments are just as seizable as the land. Wife 2 will make a payment only on condition that 1 not press her own claim against 2, so for all purposes 1 is selling that claim to 2. So just as the Buyer has a right to the land, he has a right to the proceeds of a sale of a claim on it. If side payments can be seized then coalition 12's benefits must stay with 2 and the correct model is an NTU game. (Although we are convinced by this argument, those who are still curious about the Shapley TU-value can refer to the calculations of the Shapley NTU and Maschler-Owen values in the next section, which coincide with the Shapley TU-values for these games.)

3. NTU bargaining models

An NTU game is a pair (N, V) where N is a set of n "players" and $V(S)$ is a set of vectors in $\mathbf{R}^S$ for $S \neq \phi, \subseteq N$. The set $V(S)$ is interpreted as the payoffs for the options achievable by S should it form. In the 3-cycle, $V(12)$

includes $(0, 1, -)$, and since the standard definition allows players to dispose of utility, 12 can get anything in the larger set $\{(x_1, x_2, -)|x_1 \leq 0, x_2 \leq 1\}$. (The "-" notation is a reminder of who is in the coalition.) For $A \subset R^S$, A^* is defined as A's comprehensive closure, $\{y \in R^S|y \leq x$ for some $x \in A\}$, so that $V(12) = \{(x_1, x_2, -)|x_1 \leq 0, x_2 \leq 1\}^*$.

Constructing a characteristic function

The appropriate values for a characteristic function are not obvious, and will be partly arbitrary. For the 4-tournament of Figure 1(c) it is natural to set $V(34) = (-, -, 0, 0)^*$ since wife 2 can seize the land no matter whether 4 tries to keep it himself or transfers it to 3. What about $V(13)$? If 24 seizes the land from 13, 13 can take it back but only temporarily. Assigning $V(13) = (0, -, 0, -)^*$ ignores 13's extra power compared to 34. Still this will be part of the definition of V; there are reasonable alternatives but reducing a sequential problem to a characteristic function always means discarding some information.

The NTU characteristic function follows from a three-stage scenario. Firt the coalition containing the current holder can transfer the land to any of its members not subject to a claim from the rival coalition (including possibly himself), with the receiver possibly making side payments to others in the coalition who are not subject to an outside claim. If this is impossible because everyone in the holder's coalition receives an outside claim then the land goes to the rival coalition, which can use the same tactic, i.e., transfer it within itself accompanied by side payments, the beneficiaries being anyone free from an outside claim. If that is impossible in the rival coalition too, then all players get at most zero. In summary, for $\{S, NS\}$, not both empty, specify S as the coalition containing the initial holder h, and let $K_S \subseteq S$ be the players of S who do not receive an arrow from the opposing coalition, i.e., $K_S = \{i \in S| \nexists j \in NS, j \rightarrow i \in E\}$.

The vector $(0, \ldots, 0)$ of length $|S|$ is denoted $\mathbf{0}^S$.

Stage 1. If $|K_S| \geq 1$ then $V(S) = \{x \in R^S| \sum x_i = 1$ for $i \in K_S, x_i = 0$ for $i \in S - K_S\}^*$, and $V(N - S) = \{0^{N-S}\}^*$for $N - S \neq \phi$.

Stage 2. If $|K_S| = 0$ and $|K_{N-S}| > 0$, then $V(N - S) = \{x \in R^{N-S}| \sum x_i = 1$ for $i \in K_{N-S}, x_i = 0$ for $i \in N - S - K_{N-S}\}^*$, and $V(S) = \{\mathbf{0}^S\}^*$ for $S \neq \phi$.

Stage 3. If $|K_S| = |K_{N-S}| = 0$, then $V(S) = \{\mathbf{0}^S\}^*$ for $S \neq \phi$ and $V(N-S) = \{\mathbf{0}^{N-S}\}^*$ for $N - S \neq \phi$.

Note that (1) implies $V(N) = \{(x_1, ..., x_n) | \sum x_i = 1, x_i \geq 0\}^*$.

The three-person line problem has the following characteristic function. (The coalition $\{1,2,3\}$ is here abbreviated 123, and $(1,-,0)^*$ refers to the comprehensive closure of the set containing this vector. The notation $(x_1, x_2, -)^*$ refers to the comprehensive closure of the vectors satisfying $\sum x_i = 1, x_i \geq 0$ for $i \in S$.)

$$\begin{array}{cc} V(1) = (0,-,-) & V(23) = (-,0,1) \\ \mathrm{V}(2) = (-\ ,\ 0,\ -) & \mathrm{V}(13) = (1,\ -\ ,\ 0) \\ \mathrm{V}(3) = (-\ ,\ -\ ,\ 0) & \mathrm{V}(12) = (\ \mathrm{x}_1, x_2, -) \\ \multicolumn{2}{c}{\mathrm{V}(123) = (\ \mathrm{x}_1, x_2, x_3).} \end{array}$$

The three-person cycle has the same characteristic function except that $V(12) = (0,1,-)$. For the four-person tournament,

$$\begin{array}{cc} V(1) = (0,-,-,-) & V(234) = (-,0,0,1) \\ \mathrm{V}(2) = (-\ ,\ 0,\ -\ ,\ -) & \mathrm{V}(134) = (1,\ -\ ,\ 0,\ 0) \\ \mathrm{V}(3) = (-\ ,\ -\ ,\ 0,\ -) & \mathrm{V}(124) = (\ \mathrm{x}_1, x_2, -, 0) \\ \mathrm{V}(4) = (-\ ,\ -\ ,\ -\ ,\ 0) & \mathrm{V}(123) = (0,\ \mathrm{x}_2, x_3, -) \\ \mathrm{V}(12) = (0,\ 1,\ -\ ,\ -) & \mathrm{V}(34) = (-\ ,\ -\ ,\ 0,\ 0) \\ \mathrm{V}(13) = (0,\ -\ ,\ 0,\ -) & \mathrm{V}(24) = (-\ ,\ 0,\ -\ ,\ 0) \\ \mathrm{V}(14) = (1,\ -\ ,\ -\ ,\ 0) & \mathrm{V}(23) = (-\ ,\ 0,\ 0,\ -) \\ \multicolumn{2}{c}{\mathrm{V}(1234) = (\ \mathrm{x}_1, x_2, x_3, x_4).} \end{array}$$

Noted NTU solutions include those of Harsanyi, Shapley, Maschler and Owen, Kalai and Samet, and different extensions of the TU nucleolus. Hart (2004) compares the first three and gives helpful tips for their calculation. These solutions depend on an n-vector of weights assigned to the players, interpretable as the virtual rates for utility transfer among the players. The Harsanyi, Shapley and Maschler/Owen values restrict the weights endogenously as part of calculating the allocations, but they sometimes allow multiple solution vectors. Some of these are degenerate in that they follow from giving certain players a weight of 0, so we will add the constraint that all players receive a strictly positive weight. It will turn out that because of the shape of $V(N)$ these solutions admit only the weight vector $(1, 1, \ldots, 1)$. This result reflects the fact that the land is freely transferable at these trade-offs within the grand coalition. Applied to our games Maschler and Owen's value coincides with Shapley's because all coalitions have feasible sets whose boundaries are unit simplices (although some are of reduced dimension.) Also, by chosing the unit weight vector for Kalai and Samet's solution it becomes coincident with Harsanyi's [Proofs].

Concerning a nucleolus-type NTU solution, although no single extension has become dominant (Kalai 1975; Klauke 2002), again the transferability of land in the grand coalition suggests using the 1:1 transfer rates so that we will simply sum the payoffs within coalitions to calculate excesses. In the end there are only three numerically different solution concepts, which we label Ψ_{SHP}, Ψ_{HAR} and Ψ_{NUC}. Results for the games of Figure 1 are shown there.

For the three-person line, Harsanyi's value gives a reasonable division: wives 1 and 2 split the land equally. One can imagine 2 threatening to go into a coalition with 3, thereby leaving the land in 3's hands and cutting out 1. Wife 2 might thereby induce 1 to give her a share. In Harsanyi's value membership in each coalition gives each player a "dividend," which may be negative but must be the same across all the members. In the three-person line only the coalition 12 can give positive dividends since the dividend in 13 is limited by 3's maximum payoff of 0, as is 2's maximum in coalition 23, so in the end only 12 matters.

The Shapley NTU solution for the three-person line confers a positive amount on player 3 even though he apparently has no threats or moves of any kind. A justification might be that if player 2 carried out her threat against 1, the beneficiary would be 3. He may be a bystander, but like a Swiss banker he can receive the land from 2 and hold it and so arguably he should receive something. Also, it is typical of the Shapley value to give high consideration to the grand coalition (compare Hart's, 1985, and Aumann's, 1985, respective axiomatizations of Harsanyi's and Shapley's values.)

Figure 4 shows how differently the solutions behave for other claims problems. For the 4-person line, Figure 5(a), Harsanyi gives player 4 nothing since 4 cannot add to the options of any coalition given the condition that everyone must share the benefits equally. Shapley, on other hand, cuts out Player 2 on the grounds that 2 cannot increase the total value of any coalition. Total value is the important criterion since the solution implicitly adds utility gains at ratios determined by the gradient of $V(N)$'s boundary.

FIGURE 4 HERE

As Figures 4(f) and 4(b) suggest, for Shapley's solution adding more players each with a demand on the holder helps the holder. This seems odd, but the rationale is that holder can combine with any of the new parties to transfer the land before the outside demands are pressed, and more possible partners strengthens his bargaining position.

Finally, consider a tournament of three-wives-plus-a-buyer like Figure 4(c) but where both the first second wives have signed away their claims. In the earlier case, Shapley's shares were $(5, 3, 1, 3)/12$, but here they are $(3, 1, 3, 5)/12$. The buyer gains and second wife loses, as one would expect after she reversed her arrow. Also the first wife loses and the third wife gains, a result rationalizable on the grounds that they have claims on fewer people and more powerful people, respectively.

Conjecture: (probably easy to prove) Restricted to 1:1 transfer rates, the solutions for these games for each method are unique.

4. A strategic game of sequential offers

A sequential-offer game runs as follows. At times $t = 0, 1, 2 \ldots$ the land can either pass from one player to another through a seizure or it can be divided permanently by an agreement. The claimants on the initial holder are called the *initial claimants*, and if there has been no agreement we identify the *current claimants* as those players with a claim on the current holder. If there are no current claimants the land stays with the holder forever, but otherwise one of them is selected equiprobably and this player, the *current proposer*, offers a *proposal*, a specific division of the land among the players. After the current proposer announces the proposal, all parties simultaneously announce whether they accept the proposal. If all accept, that division becomes their permanent shares, but if anyone refuses, the proposer either leaves the land with the holder or seizes it and becomes the holder. The decision to seize may depend on who has or has not accepted. (The motive for sometimes seizing and sometimes not is apparent from the three-person line, where player 2's power derives from threatening 3 with seizing and threatening 1 with not seizing.) From the next set of claimants, possibly the same or different, a proposer is selected, independently of past events. Players aim to maximize the limit of their average payoffs as $t \rightarrow$.

By defining the set of states as the pairs $\{(i, j | (i \rightarrow j) \in E\}$ comprising a holder and a proposer, we have a stochastic game (Vieille, 2002). In a stochastic game the next state depends only on the players' joint choices at the current stage and a chance event, possibly dependent on their current choices. A strategy is *Markov* if it depends only on the current state, and a Nash equilibrium is *Markov* if it involves only Markov strategies, i.e., does not depend on the current time or the previous history.

For a problem with graph $G = (N, E)$ a *path* from i to j is a sequence $(i_0, i_1, \ldots, i_k)$ such that $i_0 \to i_1, i_1 \to i_2, \ldots, i_{k-1} \to i_k \in E$ with $k \geq 1, i_0 = i$ and $i_k = j$. We assume G has a path from the initial holder to every other player, since otherwise some player would have no possible moves, would receive 0 in every equilibrium and could be eliminated. The states of the stochastic game are then equivalent to the edges E. A Markov strategy for player i is a triple $(Proposal_i, Seize_i, Accept_i)$. $Proposal_i$ is a function from E to the simplex of n-tuples x with $x_i \geq 0$ and $x_i = 1$, interpreted as the proposal i makes when he is the current claimant on j. $Seize_i$ is a function from E to the subsets of $2^{N\{i\}}$, interpreted as the list of patterns of refusals that trigger a seizure. Finally, $Accept_i$ is a subset of $\{\{j, k\} \times Proposal_j | \{j, k\} \in E, i \neq j\}$, and is interpreted as the set of proposals that i will accept.

One Markov equilibrium is the *greedy equilibrium*, where each proposer demands 1 and threatens to seize if anyone refuses, while each non-proposer refuses any offer less than 1. For the three-person line, the greedy equilibrium has the land move to player 2, then to 1, then stay there; in the three-person cycle, each gets 1/3.

Theorem 1. The greedy equilibrium is a subgame perfect equilibrium for any claims problem; its expected payoff vector is the Markov solution (Section 2).

The *agreement equilibria* are those subgame perfect Markov equilibria where the first offer is accepted. For the 3-person line, one group of agreement equilibria has the initial proposer 2 offering (0, 0, 1) and the others accepting. This is backed by 2's not seizing after any refusal. Another group involves 2 offering (1, 0, 0) and 2 seizing if 3 refuses. A third set involves 2 offering (0, 1, 0), not seizing if 1 refuses but seizing if 3 refuses. All of these moves by 2 are optimal since if 2 were to seize, 1 would offer (1, 0, 0) at the next stage.

The last group of equilibria strikes us as the most persuasive. Had the game not allowed agreements, 2 become a selfless arbitrator, allocating the prize between 1 and 3 (perhaps probabilistically) but gaining nothing. Since agreements are possible, 2 can be a corrupt arbitrator, receiving payments for his choice, and chiseling the others down to zero as happens in certain markets and auctions. While 1 and 3 wish they could collude, an equilibrium need only be resistant to *unilateral* deviations - 3 chooses her moves under the assumption that 1 will agree to the proposal and vice versa for 1, so neither gains by saying no. This is equivalent to an assumption that, having maximized his payoff, 2 lexicographically adopts the goal of frustrating a player who refused his offer. We define a *agreement solution* as a vector of payoffs of a subgame perfect Markov equilibrium in the above game, which, at the first selection of a proposer, are optimal for each proposer. Figures 1 and 4 give the agreement solution ("AGR") for some example games.

The following theorem narrows the possible offers that arise in agreement equilibria by giving the shares of the holder and the non-claimants, but not of the claimants.

Theorem 2. In the agreement solution, if there are no initial claimants the initial holder receives the total; otherwise it is divided in some way among the initial claimants.

Thus in the three-person line and the cycle, the agreement solution assigns the full payoff to player 2. Figure 4 shows a problem where the initial claimants share it unequally. If player 1 or 3 are chosen as initial proposers they take it all, but if player 2 is chosen, he offers 1 and 3 amounts equal to their expectations should the game go to another round of proposals.

5. Monotonicity

We discuss how the different allocations respond to changes in the problem. They are denoted Ψ_{HAR}, Ψ_{SHP}, Ψ_{NUC} and Ψ_{AGR}.

First, if we add a new arrow it is plausible that the player at the tip should get no more than before and that the player at the tail should get no less.

Definition. A solution rule Ψ satisfies *out-claim monotonicity* if, for all claims problems $C = (N, h, E)$ and $C' = (N, h, E \bigcup \{j \to i\})$ and all $x \in \Psi(C), x' \in \Psi(C')$, the following hold: $x_i \geq x'_i$ and $x_j \leq x'_j$.

One cannot immediately conclude that any particular NTU solution satisfies out-claim monotonicity since the three NTU solutions are not in general monotonic with the characteristic function. For two players they reduce to the Nash bargaining solution, which does not satisfy this property (Kalai and Smorodinsky 1975). The problems considered here, however, are restricted in that the grand coalition achieves the unit simplex. By considering cases one can show the following, where the new arrow is $i \rightarrow j$, the resulting characteristic function is V', and S is a coalition containing neither i or j, and

$$V(S) = V'(S),$$
$$\text{V(S + \{i\} + \{j\}) = V'(S + \{i\} + \{j\})},$$
$$\text{V(S + \{i\})} \subseteq V'(S + \{i\}),$$
$$\text{V(S + \{j\}) V'(S + \{j\})}.$$

Then one can argue that, as pointed out above, Ψ_{SHP} corresponds numerically to the Shapley TU value, whose monotonicity in this sense was proved by Young (1985), and that Ψ_{HAR} is identical to Kalai/Samet's monotonic solution with unit weights, whose monotonicity was established by those authors (1985). A comparison of Figure 4(e) and 4(f) shows that the nucleolus rule is not monotonic. In summary,

Theorem 3. The solution rules Ψ_{HAR} and Ψ_{SHP} are out-claim monotonic; Ψ_{NUC} is not.

Definition. A solution rule Ψ satisfies *in-claim monotonicity* if, for all claims problems $C = (N, h, E)$ and $C' = (N, h, E \bigcup \{i \rightarrow j\}$ and all $x \in \Psi(C), x' \in \Psi(C')$, the following hold: $x_i \geq x'_i$ and $x_j \leq x'_j$.

ADD A PARAGRAPH AND THEOREM 4 ON THIS AXIOM

Another plausible criterion is that moving the land from one player to another never helps the former and never hurts the latter.

Definition. A solution Ψ satisfies *possession monotonicity* if for all claims problems $C = (N, h, E)$ and $C' = (N, h', E)$ and all $x \in \Psi(C), x' \in \Psi(C')$, the following hold: $x_h \geq x_{h'}$ and $x_{h'} \leq x_{h'}$.

By considering certain cases we can establish some equality and inclusion relations between the characteristic functions. If V' is the game with the holder moved from h to h',

$$V(S) = V'(S) \text{ if } h, h' \notin S \text{ or } h, h \in S,$$
$$V(S) \subseteq V'(S) \text{ if } h \notin S, h' \in S$$

$$V(S)V'(S) \text{ if } h \in S, h \notin S$$

One can then apply the results of Young, Kalai/Samet to show the monotonicity of Ψ_{HAR} and Ψ_{SHP}. The sequential bargaining procedure Ψ_{SEQ} violates monotonicity, however, since with three claims in a line as in Figure 1(a), moving the holder from 3 to 2 changes 2's payoff from δ to $1-\delta$, and so harms 2 for $\delta > 1/2$.

Theorem 5. Solution rules Ψ_{SHP} and Ψ_{HAR} satisfy possession monotonicity; Ψ_{SEQ} does not.

It seems appropriate that the strategic model should *not* satisfy possession monotonicity. In the three-person line, 2's only threat against 1 is based on not possessing the land. Generally the different solutions' monotonicity or lack of it do not reflect the relative plausibility of the strategic versus coalitional game approach, since the assumptions of play differ. In the strategic approach players attach value to holding the land even temporarily, in contrast to the coalitional approach. The latter let them transfer it to a coalition partner, which was not part of the strategic approach.

6. Conclusions

The Talmud's problem is interesting not only on its own and for its implications for the coalitional and strategic approaches to bargaining. Nash had viewed the two approaches as "complementary" (Serrano, 2005), but over the years and especially in the United States coalitional models have come to often be seen as innately flawed, worth considering only until a strategic version arrives. This follows the so-called "Nash program," which Harsanyi endorsed in 1980 in these words,

> The natural remedy [for problems of NTU solutions raised by Roth] is to define the solutions for cooperative games by means of suitable bargaining models having the nature of non-cooperative games in extensive (or sometimes in normal) form. Of course in order to use this approach, one must have a mathematical criterion that will always select one specific equilibrium point of one's bargaining model as the solution. Once such a theoretical framework is available, then difficulties like those pointed out by Roth will disappear.

He added that his own tracing procedure gave a reasonable solution to Roth's example.

Harsanyi hoped for a persuasive single-valued solution but that now seems too optimistic. Even when the Nash equilibrium shares are unique, as in the examples here, the result can be counterintuitive. In the Talmud's three-person player 1 gets 0 and player 3 gets $1-\delta$ if both accept 2's offer, but refusing would guarantee these minimum amounts and possibly give them more. This solution payoffs seem irrational but the fault cannot be laid with the particular model of offers since the payoffs seem unreasonable even within that model. The process gives the floor to player 2 to make a proposal and the other two player stay silent, but even without communication they will surely realize that they can only do as well or better by turning 2 down. The difficulty is that the standard Nash equilibrium resists only deviations by one player at a time, which in this case assumes that player 1 and player 3 do not deviate together. The concept is unsuitable for n-person games where coalitions are advantageous. This difficulty has come up in past contexts (e.g., Bernheim, Peleg and Whinston, 1987:4), but the Talmudic problem provides an especially simple example.

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