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THE UTILITY OF MATHEMATICAL CONSTRUCTS IN BUILDING ARCHAEOLOGICAL THEORY

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INTRODUCTION[1]

In his book, *Analytical Archaeology*, Clarke (1968:512-513) noted three ways in which mathematical concepts are relevant to what he calls archaeological ideology:

"(i) The need for entitation and quantification ... using ... descriptive statistics";

"(ii) The need to handle relationship concepts ... using analytical inductive statistics";

"(iii) The need to handle the regularities in complex data in terms of isomorphic systems of symbols arranged in axiomatic schemes, models, or calculi"

A similar theme was iterated by Cowgill (1986:369) in a review article titled. *Archaeological Applications of Mathematical and Formal Methods.* There he referred to three broad categories comprised of "archaeological observations, analytical methods, and sociocultural theor", but then observed that although "some theory is expressed directly in mathematical terms ... the vast majority of archaeological uses of mathematical and formal techniques pertain to the domain of analytical methods or to the design of data collection". And in a recent text, *Anthropological Archaeology,* Gibbon (1984:383), though espousing the value of formal and axiomatically expressed theory in archaeological reasoning, bluntly commented that "No theory within archaeology has ever been formalized". Diametrically opposed conclusions can be drawn from these comments:

(1) the lack of formalized theory is indicative of the immaturity of archaeology as a science, or

(2) formalized theory is largely irrelevant to the development of archaeological theory.

The intent in this review is to show that lack of substantive, axiomatic-like theories in archaeology is neither inherent to the discipline, nor to the capabilities of the discipline's practitioners, nor to the alleged irrelevancy of axiomatically framed arguments for an archaeologically based theory (Salmon 1982). Rather, mathematically based techniques in the form of statistical methods and modeling have already been well-established as an essential part of archaeological data analysis (Read 1989). What is lacking, though, is application of mathematical formalism to the theoretical issues of archaeology, despite recognition of the value of axiomatically or formally expressed theory as shown in the above quotes. I suggest that the disparity between (1) the acceptance of statistical methods and (2) the lack of application of mathematical formalism stems from inadequate understanding of the way mathematics provides not only a language for the expression of relationships, but also a means for reasoning about their consequences, hence a language for extending archaeological reasoning.

To develop this argument, I will first consider the nature of mathematics and its several roles when serving as a language and conceptual system for expressing relationships and processes responsible for the structure found in data. Then I will examine several published applications of mathematical formalism directed towards the understanding of processes.

The first topic will make a fundamental distinction between mathematical methods used to express idealized patterns surmised from data and mathematical formalism used to represent process as the producer of pattern. The argument will be guided both by the philosopher C. Peirce's view of mathematics as a system for reasoning about what must be true in a hypothetical state of affairs (Peirce 1936), and by the way reasoning enters into a schema for scientific argumentation used by the philosopher Suppe (1972) in his trenchant critique of the logical positivist's view of scientific, explanatory arguments.

The second topic will consider examples -- information diffusion, dynamical systems, Catastrophe 'theory, linear programming, and marginal cost analysis -- of applications of mathematical formalism to the modeling of process.[2] These examples will highlight the restricted sense in which the reasoning power of mathematics has been brought to bear on the deeper, substantive issues of archaeology, particularly those relating to the formation, evolution and change of societal structures and systems. I will also suggest that too much reliance on assumed or claimed behavioral regularities, such as rationality, as the basis for constructing theories fail to take adequately into account the effects on behavior of social context (Keene 1985) and of the meanings provided by a culturally defined "reality".

Mathematical formalism, then, provides the means to express hypothesized processes and relationships in a manner that allows for the deduction of their implications for behavior viewed as the source for the empirical patterns distinguished and isolated by the archaeologist. As it is increasingly perceived that the logic of archaeological arguments can be expressed, developed and scrutinized through mathematics taken in Peirce's sense as a system for reasoning with schemata, then the foundation for Clarke's (1968) vision of an analytical archaeology will be set in place.

PART I: MATHEMATICS AS A SYSTEM FOR REASONING

AXIOMATIC REASONING

Both directly and indirectly, archaeologists have made extensive use of ideas and methods in applications that derive from mathematical theory. These applications have run the gamut of statistical methods and are almost as varied in kind as is their quality (see critiques by Thomas 1978; Clark 1982; Scheps 1982). The aim in this part, however, is not to criticize the details of specific applications but to stand back and to consider what constitutes mathematical thinking and how this can relate to archeological reasoning. Paradoxically, the relationship is not seen easily in these applications.

In applications, one is typically concerned with developing or applying a model through using a mathematical idiom as a means for expressing relationships thought to characterize the data in question. In contrast, mathematical thinking is concerned with the conceptual system for which the model is to be an instantiation. And while much of the mathematics that has been applied in archaeology and the other social sciences has to do with quantities, quantity, per se, has little to do with mathematics. Rather, as the philosopher Charles Peirce (1936) noted, "Mathematics is the study of what is true of hypothetical states of things" (p. 1775), based upon "reasoning with specially constructed schemata" (p. 1777), characterized by "the extraordinary use it makes of abstractions" (p. 1777) and "can have no success where it cannot generalize" (p. 1778).

The notion of mathematics as a means to draw out what must be true in a hypothetical system through using and manipulating symbols to convey and represent abstract concepts and properties has been most extensively developed in the axiomatic method. The general aim of the axiomatic method is to determine the properties logically entailed by a set of axioms, primitive terms, and definitions. While one might appeal to real-world experience for motivation of one's choice of primitives, axioms, and initial definitions, the

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connection with the real-world is neither necessary nor a criterion for evaluation of an axiomatic system.

The power of the axiomatic method can be seen through the role it has played in establishing a foundation for an extraordinary reformulation of mathematics by members of the so-called Bourbaki School of French mathematicians who view all of mathematics as a hierarchy of axiomatically defined structures (Kothe and Ballier 1985:506). The axiomatic method begins with a set of primitive terms and/or symbols whose meaning is external to the structure being defined, and a set of axioms that define relationships amongst the primitives. The axioms are taken as true and then the logically necessary conceptual structure is developed by determining other, conditionally true statements, or theorems and propositions, entailed by the axioms. New concepts are introduced and defined using only the primitives, axioms, and derived theorems and propositions for the terms and relationships which appear in the definitions.

The end result is a structure of relationships amongst symbols and primitive terms whose meaning is to be found in the properties of the abstract system so constructed, not in its relationship, if any, to real world phenomena. It can be likened to a cultural construct in the sense Leslie White has used this notion for a symbol system, but differs from more familiar cultural constructs through explicit identification of the symbolic foundation upon which it is constructed and verified. In an axiomatic system, a clear separation is made between meaning induced through the properties of the abstract system, and meaning as derived through implementation.

A classic example of an axiomatic system is provided by the axiomatization that was made of arithmetic in the 1800's by the Italian mathematician, Peano. The axiomatization is based on a series of axioms abstracted from the properties of the counting, or natural, numbers, i.e., the numbers 1, 2, 3, .… The primitives, in addition to standard mathematical symbols, are the terns 'number' and 'successor' and the symbol ' 1'. The axioms are:

(1) there is a number, called 1;
(2) to every number, n, there corresponds a unique number, n', called its successor,[3]
(3) 1 is not the successor of any number,
(4) if the numbers n and m have the property that $n = m$, then the successor numbers, n' and m', respectively, have the property that $n' = m'$; and
(5) if P is a proposition about numbers where P is true for the number 1, and, whenever P is true for the number n it is also the case that P is true for the successor number n', then P is true for every number m (Induction Axiom).

From these axioms may be defined, and then proved to be unique, the notion of addition and multiplication of the natural numbers. From addition and multiplication of the natural numbers may be defined and derived the complete system of real and complex numbers (see, for example, Pickers and Gorke 1986), the foundation for the branch of mathematics known as analysis, and through analysis the basis of mathematical models using relationships of measures of quantity, the latter being the aspect of mathematics commonly found in applications.

We see in this hierarchy the paradox mentioned above. The aspect of mathematics which has the greatest immediate importance for the archaeologist in mathematical applications, namely methods for developing and expressing models based on measurement of quantities, is not the foundation for mathematical thinking, whereas what has greatest importance in mathematical reasoning, namely the development of a -- conceptual -- system from a postulated set of axiomatic relationships, has had the least importance to the archaeologist. Nonetheless, the axiomatic method of reasoning can provide a model for the kind of reasoning that can be developed through formally expressed archaeological theory.

Elsewhere (Read 1978), I have discussed the potential of using the axiomatic method as a guide for formally expressing concepts used in archeological arguments. This discussion was criticized by Salmon (1982) for advocating a method which, supposedly, is inappropriate for archaeology. Her discussion, however, misconstrued the intent of my example which was to take seriously the assertion of the "new archaeology" that theory, in the sense of the hard sciences, can and should be developed within archaeology, and to examine how one might go about developing an axiomatically expressed theory based on concepts relevant to archaeology. A more useful criticism would have been to take up the question of whether or not the domain identified by many archaeologists as the focus for theorizing, namely regularities in behavior as inferred from archaeological data, can be axiomatized effectively.

The problem arising with axiomatization is not whether archaeologists have developed theory re-castable in an axiomatic fashion -- for one can develop theory and its axiomatic expression in roughly parallel fashion as was done, for example, with transformational grammars -- but whether there are principles or relationships suitable for restatement as axioms for an axiomatic construction. By this I mean that a particular axiomatic system can be no better than the axioms upon which it is founded. because it is fundamentally a means to carry out the logic entailed by the axioms. The axiomatization of arithmetic established that, while arithmetic developed historically out of experiences with quantities in the real world, the properties of arithmetic as a system of thought are not merely a symbolic codification of these experiences but are based on an abstraction that supersedes them.

The basic abstraction has to do with the notion that the process of going from a quantity to a successor quantity can be extended indefinitely, regardless of whether or not there are empirical sets corresponding to these quantities. A counting system such as 'one, two, three, many' -- or its modern counterpart, 'one, two, three, ..., infinity', where 'infinity' has the folk meaning of an extraordinarily large number -- is consistent with viewing sets of objects as having quantities, but not with arithmetic. In arithmetic, there is necessarily a successor to the number 'two', whether or not it is named or has ever been encountered; that successor has a successor and so on without end, according to Axiom 1. The indefinite extension of the counting numbers does not come directly from experience since all experience is finite in its extent, but as an abstraction from experience as expressed in Axiom 1.

Arithmetic, as a conceptual system, thus transcends experiences derived from dealing with quantities through being built upon, and internally consistent with, its founding concepts or axioms. In Peirce's terms, it is an assertion about a hypothetical state of affairs, namely that there are things called numbers and that every number has associated with it another number called its successor, with the structure of what we call arithmetic being the property such a system must inevitably have. The inevitableness is both the strength and weakness in the application of the axiomatic method. It is a strength because it establishes what properties are explainable as aspects of the system, and what properties require explanation as defining the system, hence the axiomatic formulation is informative of what properties are accounted for by the axioms when identification is made with real world phenomena.

The abstract, axiomatic system is linked to the real world through identification of the terms of the axiomatic system with real world properties. Arithmetic works in the real world because our empirical notion of quantity in the sense of the number of objects in a collection of objects has properties isomorphic to the primitive notions and axioms of arithmetic taken as a conceptual system. Thus, the idea of '1' corresponds to the empirical observation that there are individuated 'things'; a 'successor quantity' is expressed by the idea that the quantity of objects in a collection can be changed through augmenting the collection with another object of the same kind; a beginning 'number' by the fact that quantities begin with a collection having at least a single instance of an object and this collection is not obtainable by augmenting any other collection of objects, etc. (Read 1987). Through this identification of axioms with real world experience we create the illusion that arithmetic is coterminous with empirical reality rather than seeing it for what it is, an abstracted, conceptual structure made isomorphic with real world experiences through identification of the abstract notion of number with the experiential notion of quantity.

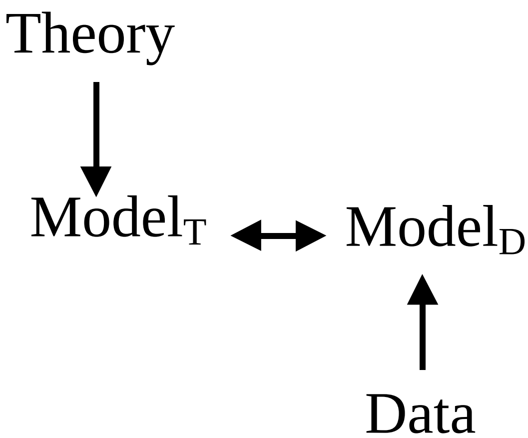
The latter is also the weakness of axiomatic systems. While axiomatic systems are relatively easy to formulate as purely mental constructs, formulating systems which will have broad ranging, useful isomorphism with real world relationships is more problematic. One domain where there has been notable success is physics.

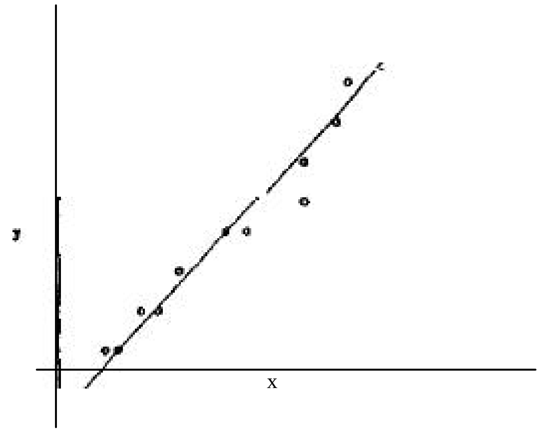
In physics the degree of identity between the formal construction and the properties of the topic domain, namely the properties of the physical universe, has been remarkable. As the physicist Hertz commented "Within our own minds we create images or symbols of the external objects, and we construct them in such a way that the logically necessary consequences of the images are again the images of the physically necessary consequences of the objects" (1894; quoted in Hermes and Markwald 1986:6). This identity between the properties of the physical world and mathematical constructs has led to a view of mathematics not as a tool for use in investigation of properties of the universe, but as the ultimate expression of those properties: "mathematics remains the method par excellence for the investigation, representation, and mastery of nature. In some domains it is all we have; if it is not reality itself, it is the closest to reality we can get" (Kline 1985:227). The connection between mathematical formulation and physical property has been seen as bordering on the miraculous: "The *miracle* of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift we neither understand nor deserve" (Wigner 1960, emphasis added).

That mathematics should have this relationship to theories of the universe neither stems from the maturity nor the hardness of physics as a science, but from the nature of the physical universe. It is as if the physical universe is the working out of fundamental principles or laws, and what we express through mathematics are the properties of the structure entailed by these principles. In physics, mathematics is no longer the codification and symbolic expression of properties and relationships found by other means, but the source for finding these properties and relationships. It has become the means for reasoning about the physical universe through appropriate symbolic schemata, as Peirce characterized the nature of mathematics. It is here that the power of mathematics comes to the fore, not in the modeling of relationships already determined, but in providing the schema through which the logic of relationships can be drawn out and explored. It is this reasoning aspect of mathematics, I suggest, that has the greatest potential for aiding in the development of substantive theory relevant to the topic domain of archaeologists.

SCHEMATA FOR RELATING DATA, MODEL AND THEORY

In his critique of the logical positivist's framework, or "Received View", for explanatory arguments, the philosopher Suppe (1972) noted that above and beyond any specific criticism of the adequacy of the Received View was its failure to account for what scientists actually do. The Received View of scientific explanation is based on an observational and a theoretical vocabulary that implied a direct connection between observation and theory, a connection at odds with scientific argumentation. Suppe noted that minimally an additional step wherein one goes from data to an idealized representation of data in the form of a model, and from a model to a theory must be part of any account of an explanatory argument. And it is quite clear from even a superficial perusal of the archaeological literature that models and their construction play a central, intermediary role in developing arguments that attempt to grapple with the complexities of explanatory reasoning.





Model_D: $y_i = a + bx_i + e_i$, where the *ei* have a $N(0, \sigma^2)$ distribution. Validation: Compute the least squares estimates for a and b and determine if the residuals, e_i, have a $N(0, \sigma^2)$ distribution.

Figure 1. A schematic diagram showing the relationships among theory, models and data..

Figure 2. A linear regression model as a prototypic form for the expression of a Model_D.

Suppe's argument needs to be extended, for it is also evident that models play at least two distinct roles in these arguments, and separation of these two senses will clarify some of the issues about relating data to theory, and mathematical reasoning to archaeological theory. Figure 1 presents a schema which distinguishes two usages of models. One usage, labelled Model_D; incorporates Suppe's idea of a model serving as an idealized representation of concrete data. The second usage, labelled Model_T, is the notion of a model as a more concrete representation of a theory, much in the sense of Nagel's (1961) discussion of a model. The argument being made here is that neither Model_D nor Model_T, is the correct usage. Both usages are part of scientific argumentation, with emphasis sometimes on the first kind of model, other times on the second kind. The connection between data and theory is through correspondence between Model_D and Model_T, in the following sense.

When examining some corpus of data, models of the first kind may be considered until a satisfactory model is obtained. Such a model will have distilled out of the full complexity of the data a set of relations that are deemed of particular importance. While the choice of measurements or features expressed in the model is not made in a theoretical vacuum, generally the guiding theory is implicit and not the primary topic under investigation. The emphasis is on abstracting from the data some set of relationships or pattern and in providing a representation of these relationships through a model (Figure 2). For our purposes here, it will be assumed that the model will be expressed in symbolic notation.

In other cases there may be concern with trying to account for the corpus of data by making reference to a set of processes alleged to give these data their form and structure. In this kind of argument the emphasis is on the relations that are entailed by the processes (Figure 3) and a model serves to express these relationships and what observations should follow if the model, constructed in accordance with the theoretically denved relationships, is applicable (Figure 4). While the data domain provides the rationale for selecting certain processes as most important, the initial emphasis is on drawing out the consequences of these processes and secondarily on determining patterns in the data.

Connection between data and theory comes through comparison, of the two kinds of models, one constructed and validated with reference to the data in question, the other

Theorem: A set of n points may be connected in exactly $n(n - 1)/2$ distinct ways.
Proof (by induction): First show that the theorem is true for $n = 1$. If $n = 1$ there are 0 connections and $1(1 - 1) = 0$.
Second, show that if the theorem is true for $n = k$ points then it is also true for $n = k + 1$ points. Suppose the theorem is true for $n = k$ points. Consider what happens with $k + 1$ points. Select k of the $k + 1$ points. These k points may be connected in $k(k - 1)/2$ ways (because of the assumption that the theorem is true for $n = k$), and the remaining point may be connected to these k points in k ways. All together, there are $k(k - 1)/2 + k = (k + 1)k/2$ connections. Hence the theorem is true for $n = k + 1$ whenever it is true for $n = k$.

By induction, the theorem is true for all n. QED

Figure 3. A theorem as a prototypic form for relationships used in the expression of a Model$_T$.

Model$_T$: For a group of n persons, there will be $n(n - 1)/2$ possible dyadic relationships.

Figure 4. Interpretation of the abstract relationship of Figure 3.

constructed and validated with reference to the asserted structuring processes. If. in addition, there is isomorphism between the two kinds of models, then one has constructed an explanatory argument for those aspects of the data that have been abstracted via Model$_D$ through the structuring processes represented in Model$_T$.

An example of Model$_D$ construction would be the well-known argument provided by Hill (1970) for the distribution of material found at Broken K pueblo; namely, that the dichotomous distribution of room sizes at Broken K, and hence their content in terms of artifacts found by the archaeologist, resulted from two basic activities -- storage and cooking/living -- which had both differential architectural expression and associated artifacts. According to the argument, if one set of rooms were the locus of cooking/living activities, and the other set of rooms the locus for storage of raw foods and the like, then artifact remains should be differentially distributed with pottery sherds from pots and vessels associated with cooking predominant in the former kind of room and pottery sherds from vessels more suitable for storage predominant in the latter kind of room. Factor analysis, with the rooms as the units and pottery types as the variables, was used to determine if the frequency of pottery types could be accounted for by the two postulated activities and if so, whether the activities were distributed in the rooms as hypothesized. Hill argued in the affirmative for both questions.

The details of the argument are not of concern here, only the general form of the argument. Despite its seemingly being cast in the form of an explanatory argument, with hypothesis, deduction, prediction and verification, it succeeds primarily in being a Model$_D$. Hill's argument has two main concerns:

(I) idealization of the actual spatial distribution of pot sherds found on the site through assuming that the spatial distribution of sherds is due to an underlying pattern produced by the location of activities themselves well segregated in space with postdepositional disturbing factors assumed only to provide unbiased noise;

(2) construction of a model that can account for the observed, idealized, spatial distribution. The model is implicitly given through use of factor analysis, which assumes that measurements made on the data are dependent variables whose values are the consequence of "factors" that serve as independent variables.

While there is reference to a potential theory aimed at accounting for the relationships among an activity, objects used in the performance of that activity, the spatial locus of the activity, the location of objects when the activity is temporally ended, the disposal of broken objects, and so on, the theory is implicit and a model for the structuring processes is assumed, namely that the spatial location of pottery sherds is essentially equivalent to the spatial location of activities. The discussion and analysis are not about the implicit theory, but focus on the data and their spatial patterning. Hence, I argue, this is essentially a concern with a Model_D wherein the primary goal is to establish an adequate model of the patterning found in the data.

In contrast, an example of a Model_T can be found in a recent paper by Johnson (1982). Johnson's argument, rephrased a bit to make it briefer, is based on the deduction that the number of dyadic relationships in a group of persons increases essentially as the square of the number of persons, *n*, in the group (see Figures 3 and 4). From this relationship he concludes that any activity which depends upon all of the dyads being activated will run into one, or possibly both, of two problems:

(1) overload of the individual's short term memory capacity which appears to be limited to about 7 distinct chunks of information being handled simultaneously;
(2) the large amount of time required to activate all the dyads.

Johnson then asserts that consensus decision making will break down and conflict will be more likely to ensue as the group size becomes larger. He uses data given by Lee (1979) on frequency of conflict in two !*Kung* groups as a test of the applicability of the model.

Johnson takes Lee's reports on conflicts amongst the *!Kung* as if these are equivalent to conflict as expressed in the model and implicitly, but erroneously, assumes that his interpretation of Lee's data is equivalent to a model of *!Kung* behavior. Essentially, he assumes that the conflict examples reported by Lee are the result of a breakdown in consensus decision which otherwise requires all or most of the dyads to be separately activated. As Read (1989b) points out, this is an inadequate model, in the sense of Model_D, for *!Kung* behavior.

Thus, in Hill's case we find construction of a Model_D, with theory and instantiation of theory through a Model_T taken as implicit, whereas in Johnson's case we find construction of a Model_T based on an argument relating the number of points and number of connections between the points, but now a Model_D of !Kung behavior is implicitly assumed.

MATHEMATICS OF PATTERN VERSUS MATHEMATICS OF RELATIONSHIPS

These two examples illustrate nicely a critical difference in the kinds of mathematical arguments that are utilized in the two kinds of models. In Hill's case the problem is largely one of inferring a pattern in the face of extraneous noise and the technique used is statistical. In Johnson's case the question has to do with the relationship between the number of persons and the number of dyads, and from this relationship, how the likelihood of reaching a consensus is affected as the group size increases.

Common to both is the use of mathematical concepts for reasoning from one set of information to another. Hill's analysis is dependent upon statistical theory which relates, in his case, "hidden" factors to measured values for the observed variables, as well as the relationship of the parameters in the model to sample data. The specific content is irrelevant to the underlying statistical theory as the theory is concerned with the interconnections between a specified model and its parameters, on the one hand, and variability and relationships in sample data as these relate to population parameters, on the other hand. Statistical methodology provides the means by which a pattern as expressed in a set of data can be modeled. Since the underlying theory for the statistical methodology is disconnected from the specific context, it is not theory about the context, but theory about patterning and its expression in sample data, hence is limited in its range of application primarily by questions about the goodness-of-fit of a model to a population.

From the viewpoint of statistical methodology, the model need not have immediate interpretation at the level of process and may in fact have no such interpretation. In applications, statistical methodology is aimed at the correctness of a model as descriptive of a given population, not whether the population is necessarily the product of a single process nor whether the model is a mode! of that process. For example, the model

$$\log y = a + b \log x + \varepsilon,$$

in which ε is an error tern, has been used to describe the relationship between rainfall and population density for hunter/gatherers in Australia (Birdsell 1953) and Africa (Martin and Read 1973). In both cases the posited model fits the sample data, hence is a valid model for these data as a description of the relationship between rainfall and population density. The statistical validity of the model, however, neither addresses the processual question of how rainfall, as a proxy measure for resource density, becomes translated into population density (but see Read 1987 for such an argument), nor even if there is a single process that characterizes all of these hunter/gathering groups. No theory is provided to characterize the process by which environmental measures become translated into population density, hence it *is, par excellence*, a Model_D. To use a distinction made by Cowgill (1986), it is a uniformity, not a regularity; that is, it is a pattern seen in the data, not a pattern derived from first principles or axioms.

In the second example, mathematical reasoning is used to connect the number of persons in a group to the number of dyads, and the number of dyads to the likelihood of a breakdown in consensus decision making if the latter is dependent upon all of the dyads being activated. Here the reasoning is tailored to the specific process under investigation and the aim of the mathematical reasoning is to establish a general relationship among these variables. More exactly, the mathematical reasoning is aimed at establishing the relationships within the structure that has been defined, hence the argument is expressed abstractly and then given interpretation in the form of a model for the abstract argument. The interpretation depends upon giving the abstracted variables content, such as number of persons, number of dyads, likelihood of not reaching a consensus, and so on, but, in the mathematical sense, the functional form of the model is a necessary consequence of the posited relations, hence it is a Model_T. In Cowgill's terms, the Model_T describes a -- hypothetical -- regularity, not a uniformity.

Whereas a Model_D is validated through its fit to a set of data, a Model_T is validated through it correctly representing the relationships derived in the theory for which it is a model. While a Model_T may be valid for a process but not fit a set of data, a Model_D may be valid for a set of data but riot be a model for any process. Explanation requires that a Model_T also be a Model_D, but this is a necessary, not a sufficient condition. It is not sufficient since the posited process used for constructing a Model_T may be a summary kind of process and the pattern expressed through a Model_D may be an overly simplified idealization. The overall argument is explanatory in a conditional sense: *If* the stated process is sufficient *and* the idealization is not overly simplified, *then* the correspondence between a Model_D and a Model_T establishes an explanatory argument for the data at hand. Otherwise. the argument is explanatory in form but not satisfactory (Read and LeBlanc 1978).

ANALYTICAL LEVEL FOR THEORY CONSTRUCTION

The distinction between a model of a process and a model for a pattern entails another consideration, the analytical level at which the two kinds of models are aimed.

Pattern, as it relates to archaeological data, is expressed through the material remnants of behavior, whereas process underlies behavior. The question that arises is whether or not behavior can serve as a common meeting ground for both Model_D and Model_T kinds of models, or whether these two kinds of models involve different levels, with behavior the bridge between them.

To put it another way, the claim that behavior is the appropriate level for developing theory implies that theory must be couched in relationships abstractable from concrete actions, hence in relationships which are neither time nor space specific if theory is to be based on universal processes. If so, then culture becomes a kind of residual category with little direct importance in formulating appropriate theory. Yet a mathematical theory of behavior seems to demand a kind of regularity which seems unrealistic. By positing universal processes, the basis for locally differentiated behavior would demand special explanations that contradict precisely those premises.

I am not arguing that pan-human regularities do riot exist, for the notion of a species implies a shared genetic system and with that, the potential of shared biological traits, hence of universal behavioral traits emanating primarily from this biological substratum. The question is not whether there are universal behaviors, but whether these alone suffice to serve as a theory of human systems.

An alternative is to take culture in its sense of a constricted reality, and view culture as the means by which external phenomena are given meaning and interpretation, and through meaning and interpretation, the basis for behavior. For example, I have shown elsewhere (Read 1978) that *!Kung* camps apparently have a regularity in their spatial construction based on the spacing between huts. One can account for the empirical data on the relationship between number of families in a camp and the camp area by positing that huts should be at a fixed distance from one another, independent of the number of huts in the camp. Further, the distribution of families in huts is also not happenstance, but a direct mapping of kinship relations according to kin distance from the focal family of the camp (see Figures 2.1 and 2.2 in Yellen 1976).

The spacing of huts does not have a material basis in that it has no apparent connection to material aspects of *!Kung* existence, here cannot be understood without reference to culture through the fact that this particular spatial arrangement is but one out of many possible spatial relationships that could have been utilized. Through being given cultural meaning, space comes to affect and provide the context for behavior, rather than the reverse.

The spatial arrangement of families in huts according to kinship relations is even more clearly part of a constructed reality since the kinship system is a conceptual construction abstracted out of, but transcending, the biological relations of parent and offspring, much as arithmetic is a conceptual construction abstracted out of, but transcending, the quantity of objects in collections of objects as discussed above. A kinship terminology, taken as a conceptual structure, is amenable to modeling as an algebraic construction (Read 1984), hence amenable to theorizing in an axiomatic sense. Neither biology nor behavior determines kin relations, hence it is the constructed reality of a kinship universe that provides the framework within which behavior takes place, and the specific behavior cannot be understood without first understanding that constructed reality.

If behavior is the individual or group's action with respect to phenomena that have been given meaning and interpretation through culture, then an adequate theory will have to begin at the level of culture, not behavior. Or, more accurately, a theory of behavior will have to take into account the culturally constructed universe within which behavior takes place. The cultural aspects give, as it were, the initial conditions and without knowledge of the initial conditions, models of behavior will be deficient as they attempt to grapple with more than summary aspects of behavior. Mathematical formalism aimed at expressing the logic and implications of a theory of behavior will need to incorporate the properties of this culturally constructed universe.

PART II: MATHEMATICAL APPLICATIONS

INTRODUCTION

Whereas mathematics uses symbols, archaeology deals with material objects, hence mathematical applications necessarily begin with a mapping from phenomena to symbols in either a quantitative or qualitative mode. The mapping provides a symbolic representation of the material objects, which enables a suitable calculus to be invoked for reasoning about processes. The familiar procedure of measuring *m* objects via a metric measure, *V*, for example, allows for a symbolic representation of these objects in the form of a vector

$$\mathbf{V} = (v_1, v_2, \ldots, v_m),$$

where v_i is the value of the measurement V for the i^{th} object, as a point in Euclidean *m*-space, a finite dimensional vector space. This representation permits a claim such as: "one measurement, *V*, is causally related to another measurement, *W* ", to be symbolically expressed, say, as a linear regression model relating the vector **V** for one set of measurements to the vector **W** for the other set of measurements. Implications and properties of the postulated relationship can now be determined through reasoning based on the properties of vector spaces. For instance, the mathematical expression for a least squares solution to estimates of the *n* coefficients, with $n < m$, the number of objects being measured, in a linear model relating the measures *V* and *W* is determined through orthogonal projections of vector spaces, specifically through projection of the vector **W** in the *m*-dimensional measurement space onto the *n*-dimensional space of coefficients (see Leon 1986:188-197).

Thus, as has been discussed by Borillo (1977), archaeological problems may be studied by using a symbolic representation to embed them into a domain wherein the logical implications of relationships can be developed. In addition, a specific solution may depend upon positing certain assumptions, e.g., that the relationship in question is linear and parameters are to be estimated using a least-squares solution. The symbolic representation allows for implications and/or computational methods applicable to the empirical domain to be constructed through properties derived by reasoning about properties true of the symbolic system, e.g., the properties of orthogonal projections of vector spaces. Finally, the implications and/or methods are translated back to the empirical domain for their implementation.

In applications, the underlying theory for the computational method, such as the orthogonal projections in the above example, is generally taken as a given. This, however, is but a convenience that is satisfactory only for as long as there is no discrepancy between the underlying theory and the assumptions used to formulate a particular method, on the one hand, and the relationships as perceived by the archaeologist, on the other hand. An assumption such as a linear relationship may, however, be unrealistic or the presumed independent/dependent relationship of variables may be untenable when translated back into what it means in the archaeological context. When the assumed congruence between method defined through the symbolic representation and data structure as considered by the archaeologist breaks down, then the reasoning used to link method and data through the symbolic representation must be reexamined and corrected (Read 1985, 1987). To do so requires that the mapping from material object to symbolic representation be made explicit so that the conclusions drawn from symbolic/mathematical reasoning can be properly translated back to properties in the archaeological domain for confirmation.

In this second part of the review several applications of mathematical reasoning to archaeological problems will be considered with respect to their success in adequately linking (1), empirically defined relationships with mathematically defined relationships, and (2), the symbolic with the empirical domain. It will be seen that a number of deep issues regarding attempts to symbolically model properties distinguished in the archaeological domain arise when this comparison is made. These issues relate, in particular, to the ability of human systems to change and modify themselves according to goals that change through time, on the one hand, and the common assumption of relative stability of the structure of Model_T models used to express formal properties of systems, on the other hand.

Typically, modeling begins by assuming structural stability of a model's form and parameters. But structural stability does not characterize human systems. In human systems both structural form and parameters are subject to change on time scales commensurate with time scales considered appropriate for the model by the objects of the model, namely human actors. I suggest that a major challenge facing effective -- mathematical -- modeling of the human systems considered by archaeologists is to develop models that can take into account this capacity for self-modification according to internally constructed and defined goals.

MATHEMATICAL REPRESENTATION

Applying mathematical reasoning effectively to the archaeological domain, then, depends upon determining a symbolic representation in which the logical implications of processes -- whose conceptual origin lies in archaeological reasoning -- can be carried out. Viewed in this manner, application of mathematical reasoning becomes a means to extend and expand upon archaeological reasoning, though in an idiom that may be less familiar than verbally stated arguments.

Mathematical representation differs from verbal argument in its relationship to explanation, for scientific explanation essentially involves subsumption of the particular case as an instance of generalized and abstracted relationships that, in turn, may be derived from more fundamental properties via deductive reasoning. Thus, an elliptical orbit of a planet about the sun is a particular instance of one mass revolving about another, and the orbital form is derived deductively from the equation linking force, mass and acceleration, $F = ma$, and the equation giving the force of attraction of two masses, $F = GMm/R^2$. Contrariwise, when an argument is couched in a non-symbolic form, appeal is often made to seemingly reasonable arguments, not logical deductions, thereby making the connection between fundamental properties and specific instance more tenuous.

How the two forms of argumentation differ in their relationship to explanation can be illustrated with an example prototypic of many applications of mathematical argumentation to archaeology made through borrowing of models from other disciplines (see Keene 1983). The example is taken from the construction by Jacobsen and Eighmy (1980:333) of a "mathematical theory of horse adoption on the North American plains" with the goal to "understand the temporal form and rate of adoption". The means is an information diffusion/adoption model borrowed from Dodd (1953, 1955, 1958).

Verbal argument

Jacobsen and Eighmy begin by referring to a verbally expressed argument given by Wissler (1923) for a uniform rate of adoption of traits in general, and by Ewers (1955) for an increasing rate of adoption of the horse by pre-historic Plains Indians. Both of these arguments cannot be true simultaneously, and neither provides an effective explanation for why the rate should take on the asserted form (Jacobsen and Eighmy 1980). The verbal arguments lack the means to *demonstrate* a connection between the posited rate of adoption of traits by whole societies and more basic, underlying processes upon which societal adoption depends.

Further, logical implications of the posited rate for the data at hand are not explored in any detail. Indeed, it would be difficult to develop implications in a verbally expressed argument for, say, how the number of groups adopting the horse should vary through time. Yet the latter is information ore would like to have. Is there, in effect, a single adoption process operating so that particular instances of trait adoption, either by different groups or of different kinds of things being adopted by individuals and/or groups, are but differences at the level of boundary conditions and parameter values in a common model? Or does the manner in which adoption takes place possibly differ in its temporal structure across groups and/or things being adopted?

Symbolic representation

Jacobsen and Eighmy address questions of this nature by embedding the particular instance of horse adoption into a formalized theory of adoption/diffusion of ideas, traits, etc. through a population (Dodd 1953, 1955, 1968; see also Lave and March 1975). In this theory the underlying, basic process is given symbolic representation through which a deductive argument linking the underlying process with the equational form for the time-based pattern of adoption/diffusion of a trait throughout a population can be constructed. The latter may then be given empirical test for its applicability to empirical contexts.

The argument runs as follows (see Lave and March 1975). For diffusion to take place, individuals who currently have the information being diffused must be in contact with those who do not have it. Further, when in contact, there must be conveyance of the information from the one with the information to the one without, and the latter must accept the information. Next, the new recipients of the information must repeat the process with yet other individuals who currently do not have the information.

Of these several conditions, the process by which there will be an encounter between a person with, and a person without the information is primary for structuring the form of the diffusion. The encounter process can be given representation through the following probability argument.

Suppose we have N persons each of whore is in exactly one of two states: '+' or '-', where the '+' state means that the person has the information, and the '-' state means that the person does not have the information. Let $P(t)$ be the function giving the number of the persons who are in the '+' state at time t, and let $Q(t) = N - P(t)$ be the number of persons in the '-' state at time t. Call a meeting of two persons an encounter event. Let the symbol [-,-] denote an encounter event where neither person has the trait, and call it a Type 1 event; similarly, let the symbol [+,-] denote the encounter event where exactly one of the two has the trait without regard to order, and call it a Type 2 event; finally, let the symbol [+,+] denote an encounter event where both persons have the trait, and call it a Type 3 event. For there to be a change in $P(t)$, there first must be a Type 2 encounter and second a transformation from a [+,-] event to a [+,+] event. What we need to know is how Type 2 encounters and their transformation to Type 3 events are structured through time.

Suppose that the likelihood of two persons having an encounter is independent of the respective states of the two persons, hence random with respect to their respective states. If the encounter is a random event, the probability that one of the two persons is in, say, the '+' state at time t is $p(t)/N$ and that the other person is in the '-' state is $q(t)/(N - 1)$. The event, [+,-], has probability

$$\mathrm{Prob}([+,-]) = [2P(t)/N] \times [Q(t)/(N-1)] = (2N/(N-1))p(t)q(t) = kp(t)q(t),$$

where: $p(t) = P(t)/N$ and $q(t) = Q(t)/N$.

Thus we can postulate that the rate of change from persons in the '-'state to persons in the '+' state will be proportional to $p(t)q(t)$. Note that this expression also allows for less than perfect information transfer without further modification so long as the likelihood of transforming a [+,-] event into a [+,+] event is constant and independent of the value of $P(t)$ or $Q(t)$.

If the population is large enough, we may treat $p(t)$ and $q(t)$ as if they are continuous functions (but see below) and we arrive at:

$$dp(t)/dt = kp(t)q(t) = k[p(t) - p^2(t)] \tag{2}$$

as a Model_T expressing the rate at which a trait diffuses through a population when it is passed via individual, random encounters.

Note that other processes may be defined for transformation of Type 2 encounters to Type 3 events such as broadcast of information from a few individuals to many individuals (see Lave and March 1975), or of information only passing from key individuals. Yet another model would be produced if individuals with information only contacted nearby persons so that diffusion is in the form of a wave front, a model that has been used by Ammerman and Cavalli-Sforza (1971, 1973. 1979) for the initial spread of agricultural information through Europe.

At this point, the original, verbal idea has been given symbolic representation. From the symbolic representation it may be seen that the behavior of the system is not determined by what *p(t)* and *q(t)* "mean", but by how *p(t)* varies with time through transformation of Type 2 encounters into Type 3 encounters. Whether *p(t)* is the number of persons in a social group who have heard a rumor, or the number of Plains Indians groups who have horses is irrelevant in terms of the representation so long as the process by which the trait or information passes from one person to another person, or one group to another group, is adequately captured by the stipulated model for formation and transformation of encounters. What one wants to know, then, is the manner in which *p(t)* varies with time; that is, the functional form of *p(t).* Or to put it another way, What -- mathematical -- function would satisfy the above equation? That answer is found by turning from representation to deduction.

Deductive argument

Equation (2) is an example of what is called a **first order** -- no term has derivatives beyond the first derivative -- **linear** -- only powers of *p(t)* are added together -- **differential** -- due to the derivative, *dp(t)ldt,* symbolically expressing the rate of change of *p(t)* with time -- **equation**. Within mathematics there is a body of theory that has to do with the solutions of differential equations and within that theory are methods for solving particular differential equations. Equation (2) is called a *separable* differential equation since, using differentials, it can be rewritten in the form $dp/(kp - p^2) = dt$, and now all terns involving p are on the left side of the equation and all terms involving t are on the right side. The equation may be solved by integrating both sides of the equation, equating the results, and then algebraically simplifying them. After these steps are carried out, one finds that the solution to the equation is given by:

$$p(t) = 1/(1 + q_0/p_0e^{-kt}), \tag{3}$$

where: p_0 and q_0 are the proportion of persons with and without the trait at time 0. respectively.

The terns p_0 and q_0 are the boundary conditions, the initial numbers of groups with and without horses, respectively, and k is a parameter for the equation whose meaning is expressed in Equation (1). The graph of $p(t)$ is the well-known S-shaped, or sigmoid, curve which will approach p_0 asymptotically from above as $t \rightarrow 0$, and will approach 1 asymptotically from below as t increases indefinitely. In Jacobsen's and Eighmy's study, a modified form of Equation (2) which has been multiplied by N is used so that the curve is bounded above by N, the total number of persons or groups.

Fit of Model, with empirical data

The Model_T given by this equation was compared by Jacobsen and Eighmy to empirical data on dates for adoption of the horse by various Plains Indian tribes geographically separated into Northern and Southern tribes. The division into two groups reflects the assertion that the pattern of horse adoption was not the same for all Plains tribes. The Model_D used for these data consists of a cumulative graph of the number of groups who have adopted the horse versus time. Though this might not appear to be a Model_D at first glance, its idealized form for the data at hand -- which constitutes the basis for calling it a Model_D -- is demonstrated through the criticism expressed by Ewers (1981), whose data were used to produce the graph, of Jacobsen and Eighmy's claim that it is possible to date horse adoption by a Plains tribe to the nearest 5 years.

Jacobsen and Eighmy found Pearson product moment correlation r^2 values of 0.99 and 0.98 for these two groups -- a seemingly almost perfect fit between Model_T and Model_D. The conditions for an explanatory argument linking a particular observation to a general theory seem to be satisfied, and it would appear that the pattern of change through time in the number of Plains tribes who have adopted the horse has an explanation through the model used to instantiate a generalized theory for the adoption/diffusion of ideas. The only difficulty, as will be discussed in the next section, lies in the failure of Jacobsen and Eighmy to demonstrate that the process underlying the model derived from Dodd's theory is the process operating in the situation considered by them. This problem is not unique to their application; it arises repeatedly when the basis for application of a Model_T to a context is primarily by analogy and not through derivation (see Read 1985). Goodness of fit between a Model_T and a Model_D *alone* is not sufficient as it is perfectly possible to have a good fit with the wrong model as shown by Read and Read (1970) in a critique of an application of game theory to decisions made by fishing captains in a small Jamaican village.

RATIONALE FOR MODELS: MODELING BY ANALOGY

Modeling by analogy refers to the situation where a model is transferred from one domain to another on the basis of a common goal, rather than through demonstration of isomorphic processes. For example, Equation (2) was derived as a Model_T for adoption/diffusion of ideas in a population based upon the notion that ideas are transferred from one person to another through dyadic encounters produced randomly in the population; further, this model has been given empirical support in a number of domains for the validity of its application. Jacobsen and Eighmy have examined a different domain, though, namely the diffusion of the horse through Plains Indian tribes, which ostensibly also involves diffusion of ideas, but in this case from one tribe to another. So the analogy:

$$\text{person : diffusion of ideas :: tribe : diffusion of horse usage,} \qquad (4)$$

becomes the means to justify the use of the theory and model applicable to the relationship on the left side of the analogy as the theory and model for the right side of the analogy.

But analogies do not establish isomorphism between the respective processes and in this case the processes are not the same. Equation (2) was derived by assuming that ideas are transferred when there are dyadic encounters whose occurrence is random with respect to the members of the population. The analogous statement for the Plains tribes would be that tribe A which currently is not using the horse would adopt it after a random encounter with tribe B which currently is using the horse. A pattern of random encounters seems implausible given the geographic scale over which the Plains tribes were distributed and their inter-tribal networks. It would seem more likely, since tribes are relatively fixed in space, that diffusion took place between neighboring tribes. This would lead to the wave front type of diffusion discussed by Ammerman and Cavalli-Sforza (1973, 1979) in their argument for the initial diffusion of agriculture from the Middle East to Europe.

An advancing wave model would also lead to an S-shaped curve for the cumulative number of tribes who have adopted the horse through time. Consider, for example, the tribes as points in a rectangle and suppose a circle represents the wave front and expands by increasing its radius; points will be enclosed by the expanding circle at an increasing rate until the circle begins to cover most of the rectangle, at which time the rate at which points are enclosed will decrease, hence leading to an S-curve. Thus, the good fit Jacobsen and Eighmy found may only reflect the fact that an S-curve is a good Model_D for these data, and that different processes can give rise to similar empirical observations. A critical test for distinguishing between a wave front model for diffusion and Dodd's model based on random occurring dyadic encounters could be based on the temporal/spatial pattern of horse adoption by the Plains tribes.

Another shortcoming arises from the fact that the original model was predicated upon a fixed pattern for the transformation of [+, -] encounters into [+, +] encounters, and no model was developed for this transformation (but see Renfrew 1984a:396-397 for the outline of such a model). Such simplification is, of course, widespread when trying to develop theory applicable to complex situations. It arises in a mre extreme -- and certainly more controversial -- form in the attempts of some sociobiologists to reduce human cultural phenomena to the working out of essentially biological processes (e.g., Hughes 1988). Unfortunately, a mathematical representation offers no direct way to resolve this problem. It can be considered indirectly through distinguishing between properties that would logically be entailed by the posited processes and those that would not. In the context of Dodd's theory, the diffusion process is structured by the process that leads to encounters between those with and those without the information in question. What, then, of situations that differ in the pattern of diffusion, yet where the model for encounters remains essentially fixed?

This is a situation discussed by Spratt (1982:80) who considers innovation to consist of two parts:

(1) a driving force for innovation -- which entails a "'thermodynamic' innovation model [that] describes the driving force or incentive which motivates it"; and
(2) a kinetic model that describes "the speed at which the innovation in fact takes place".

The actual pattern is the balance between these two processes. Spratt notes that a "sigmoid curve *X* represents a successful innovation in which the market builds up by recommendation of one satisfied customer to a new customer" -- which is essentially the process assumed in Dodd's models -- while a linear curve might result through "a relatively unsuccessful innovation *Y* in which sales build up in proportion to the marketing effort of the innovator" (Spratt 1982:82). In the latter situation, encounters, per se, do not lead to a transmittal of information as the encounter is only the means for yet another process to operate, namely persuasion on the part of the person with the information.

The representation Spratt uses for the kinetic aspect of the process of adoption of innovation is essentially a Model_D in the form of what is called 'Critical Path Analysis', as developed by the du Pont company (Spratt 1982). The idea of Critical Path Analysis is to identify the various steps that need to be activated between initial idea and final sales, along with the likely time duration for these steps. The thermodynamic part is described through a "Cash Flow Curve", and "negative cash-flow" is -- in modem innovations -- seen as a key factor in early stages of development.

When Spratt attempts to apply the framework to non-industrial societies, the problems that arise with what are essentially verbal arguments become evident. Negative cash-flow needs to be translated into terns relevant, say, to a hunting/gathering society to account for why the bow and arrow -- ostensibly having low cost in its production and high returns -- only appeared as late as it did in human history. Despite attempts to provide a more rigorous framework, Spratt falls back on "just-so" kinds of explanations and suggests:

> "One possibility is that their world-outlook would not be conducive to this innovation. Clark (1963) tells us 'There was a deeply felt community between man and the animals he hunted for his food'. To launch a devastating projectile might have been inconceivable in this climate of thought" (Sprats 1982:87-88).

How or why such "world-outlooks" come about or change, assuming that world-outlook is the equivalent of negative cash-flow, is not accounted for, hence one is reduced to arguments that appeal to one's sense of plausibility.

Verbal arguments are also defective when trying to link one process to another at the level of what seems to be analogous outcomes. Spratt notes that the sigmoid curve implies a very rapid rate of change in the middle portion of the curve and that this is essentially the same as Renfrew's (1978) use of Catastrophe Theory (Thom 1975) "to explain the very rapid changes which can take place in material cultures" (Spratt 1982:81). While the *consequence*, rapid change in both cases, seems to be similar, the *processes* involved are unrelated. The sigmoid curve derives, for example, from Equation (2) which is based upon the behavior of a system whose properties are determined by this equation. In particular, the parameters that appear in Equation (2) are fixed. Though there is rapid change, it is continuous change deriving from change in the value of the variable, t.

In Catastrophe Theory. however, rapid change arises from discontinuities. For some differential equations, the surface defined by the set of equilibrium values (i.e., values x, that satisfy $dx/dt = 0$) as parameter values are allowed to vary, may be a surface that has been "folded over". For example, the set of solutions, $[(x_s: a, b)]$, for the condition $dx/dt = 0$ applied to the differential equation, $dx/dt = x' - ax - b$, forms a surface with an *S*-shaped fold (see, e.g., Figure 21.2 in Renfrew 1979:492; Figure 7.4 in Beltrami 1987:176). When there is such a fold, a shift due to a small change in parameter value from an equilibrium value on the upper part of the fold to an equilibrium value on the lower part of the fold is locally discontinuous. Mathematically, the change in going via a discontinuity from the upper surface to the lower one is instantaneous; in an application, it would be a "catastrophe". This rapid change in system due to a shift across a discontinuity in equilibrium values is not the same process of rapid change that occurs as one moves smoothly along the sigmoid curve.

CHANGE IN PARAMETER VALUES

The situation dealt with by Catastrophe Theory addresses a complex problem that has only been poorly considered when dealing with human systems. The problem has to do with the assumed constancy of parameter values in models for most theories, whereas human systems are capable of internally driven change and modification of parameter values.

The nature of the problems that arise with change in parameter values can be illustrated with a commonly used model for demographic growth or decay, namely:

$$dp/dt = rp(1 - ap) \qquad (5)$$

The usual interpretation of the parameters *r* and *a* is that *r* measures the intrinsic growth rate, *a* is a summary term for all factors that inhibit growth and $1/a$ is the equilibrium population value since $dpldt = 0$ when $p = 1/a$. Note that Equation (5) has the same form as Equation (2) for diffusion and is identical to Equation (2) if $a = 1$. Thus, both diffusion as modeled by Equation (2) and growth as modeled by Equation (5) are structurally similar processes and both lead to a sigmoid curve for their solution.

Eighmy (1979) has applied Equation (5) to population growth curves in the Southwest for 14 different sites, using number of logs in the construction as a proxy measure for population size, with time spans in a site ranging from 15 to 135 years. He found reasonable fit in each case, though with different parameter values (e.g., *r* varied from 0.15 to 0.81 across the 14 cases). Contrariwise, Plog (1979) examined Hay Hollow Valley over a time span of about 1000 years and found a complex pattern that could not be summarized in any simple manner. The differences between the two studies seem to relate to the fact that demographic processes, though relatively unchanging over the short run, are not invariable and are restructurable. The parameters *r* and *k* are only locally constant.

While the fact that parameters are not constant can be expressed easily in a modified form of Equation (5), namely,

$$dp/dt = r(t)p(t)[1 - a(t)p(t)], \qquad (6)$$

this formulation only defers the problem to one of identifying the functions *r(t)* and *a(t)* through an as yet unarticulated theory. And therein lies the difficulty. But before discussing this point further, it will be useful to give a brief characterization of dynamic system modeling, for which Equations (2) and (5) are specific instances.

Dynamic system modeling

There are two classes of models to be noted: continuous and discrete. It is assumed that there is some set of variables, called 'state variables', whose values at time *t* are used to characterize the state of the system at this point in time. First, consider the case where it is assumed that state variables vary continuously with time. The characterization may be motivated by noting that Equations (2) and (5) are of the form

$$dx/dt = x(t) \qquad (7)$$

where *x* is a state variable whose value, *x(t)*, represents the state of the system modeled by Equation (b) at time *t.* More generally, a system *S* may be defined to consist of:
(1) a finite set of state variables $\{ x_1, x_2, ..., x_n \}$,
(2) a set of differential equations expressing the rates of change of these variables

$$dx_1/dt = f_1(x_1, x_2, ..., x_n)$$
$$dx_2/dt = f_2 (x_1, x_2, ..., x_n)$$
$$\ldots$$
$$dx_n /dt = f_n (x_1, x_2, ..., x_n),$$

(3) a set of constraints

$$a_1 \pounds \; x_1 \pounds \; b_1$$

$$a_2 \pounds x_2 \pounds b_2$$

$$a_n \pounds x_n \pounds b_n$$

These relationships *may be* written mane succinctly in vector *notation:*
(1') A vector $\boldsymbol{x}$ of state variables,
(2') a vector differential equation, $d\boldsymbol{x}/dt = \boldsymbol{f(x)}$,
(3') vector constraints, $\boldsymbol{a} \pounds \boldsymbol{x} \pounds \boldsymbol{b}$
The vector $\boldsymbol{x}(t)$ expresses the state of the system at time t; the vector differential equation $d\boldsymbol{x}/dt = \boldsymbol{f(x)}$ expresses the rate of change of each state variable as a function of the current value of the state variables, which also implies that it is a "forgetful" system, and the constraints represent physical limitations on the values for the state variables, such as nonnegative values for population sizes.

A dynamic system based on difference equations differs from the above characterization by assuming that state variables do not vary continuously with time but only change values at discrete times t_1, t_2, ... In the place of derivatives, $dx_i\ /dt$, quotients may be used:

$$Dx/Dt = (x(t_{i+1}) - x(t_i))/(t_{\,i+1} - t_{\,i}) \qquad (8)$$

The contrast is not merely one of discrete versus continuous, but can lead to radically different properties for analogous models. For example, Equations (2) and (5) have stable equilibrium solutions but their difference equation parallel, which models more exactly the dynamics of horse diffusion examined by Jacobsen and Eighmy, exhibits highly complex behavior (Beltrami 1987:218-226) sometimes referred to as chaotic (Strang 1986:504-507).

A Model_T consists of a specification, such as Equation (2) or (5), for the functions $f_1, f_2, ..., f_n$ and in such a specification will appear parameters α_1, α_2, ..., α_m. The parameters are generally assumed to be constant and analysis typically consists of determining stability and equilibrium properties of the system S so defined (see, for example, Strang 1986; Beltrami 1987).

Applications of dynamic systems

An example of the use of a discrete dynamical system is provided by Ammerman *et al.* (1978) in their critique of the static -- or Model_D -- model used by Renfrew *et al.* (1968) for the spatial distribution of obsidian passed from one site to another through trade. The static model is of the form

$$Q = f(D), \qquad (9)$$

for the quantity, Q, of obsidian at a site at distance, D, from a source. Ammerman et al. (1978) suggest using instead a dynamic model -- a ModeI_T -- of the form

$$DQ/Dt = f(D, p, d). \qquad (10)$$

where: p =the proportion of a group's obsidian passed to the next group,
d = drop, i.e., discard, rate.

The specific model is given in the form of a difference equation:

$$Q_{t+1}(j) = [(1 - p)Q_t(j) + pQ_t(j - 1)](1 - d). \tag{11}$$

where: $Q_t(j)$ = amount at site j at time t

When $d = 0$, which is an implicit assumption underlying Equation (9), one obtains the equilibrium solution of all sites having the same quantity, whereas for $d > 0$ equilibrium values will be linearly decreasing with distance versus $log(Q(j))$. This, in conjunction with Renfrew *et al.*'s (1968) claim for a Model_D of the same form, establishes the basis for an explanatory argument (but see Read 1989 for a critique of the assumptions underlying Equation (11)).

More generally, Cooke (1979) has suggested the use of models based on dynamic systems modeling, game theory, optimization theory and graph theory as ways for the archaeologist to have "new tools for the difficult task of understanding cultural change" (p. 46). The enormity of the task is perhaps indicated by a failed attempt to apply precisely these tools to construct an "explanation of the emergence of complex societies in the Aegean in the third and second millennia BC" (Cooke and Renfrew 1979:328). The means was a simulation using a system based on six state variables relating to subsistence, metallurgy, craft, social roles, culture and external trade, respectively. A linear, structural model was defined in which initially a matrix A of interactions between pairs of variables was constructed, with '1' indicating a positive relationship, '0' no relationship and '-1' a negative relationship. A second simulation used quantitative estimates for these interactions, but qualitatively the same results were obtained: an increase in the values of variables to their maximum value or decay to a value of 0.

Cooke and Renfrew deductively derive that the system will have an equilibrium value between zero and some maximum value if, and only if, the determinant of $A = 0$, where A is the matrix of parameters for variable interaction. Since it is unlikely that A will be a singular matrix (i.e. det $A = 0$), they conclude that "these equations, in their present form, are inappropriate for models in which equilibrium states are expected" (p. 341). They conclude that failure to find equilibrium states -- which are assumed to be a characteristic of real world systems -- is due to using constants for the interaction parameters.

Because archaeologists necessarily deal with contexts that have left material remains of their existence, hence are likely to have had relative longevity, it is sometimes argued that much of what is found by the archaeologist is the remnant of systems more or less in equilibrium. If so, an alternative to modeling the system directly, as Cooke and Renfrew attempted to do, and then determining how the system converges to an equilibrium state, would be through examining the trajectory of systems as they change from one equilibrium state to another. Indeed, models based on rationality or optimality of behavior implicitly use this approach when examining how behavior will change under new conditions.

Change in parameter values and *catastrophe theory*

The effect of changing parameter values is examined in Catastrophe Theory through considering the surface defined by equilibrium states for a system such as Equation (2) when its parameters, in this case r and a, vary. As mentioned above in some cases, but not for Equation (2), the surface is folded over so that a small change in parameters values can drive the system to shift discontinuously to a new equilibrium state. A discontinuous shift to a new equilibrium contrasts with systems such as Equation (2) where a small perturbation in a parameter value only leads to a small shift in the equilibrium state.

Renfrew (1979) has used the language of Catastrophe Theory -- in what Casti (1989:175) refers to as the "metaphysical way" -- to discuss shifts from one form of political organization to another, such as from segmentary to chiefdom to state systems (Figure 21.8 in Renfrew 1979:501). Renfrew considered the topological properties of what is known as a butterfly catastrophe (Zeeman 1977) and identified portions of the surface with each of segmentary, chiefdom and state societies and concluded that a change from a segmentary to state society might involve something analogous to a catastrophe; i.e., a locally discontinuous shift from one equilibrium value to another. Renfrew recognized that the argument is primarily heuristic and that nothing has been proven "in the mathematical sense" (p. 504). As in Spratt's paper, Renfrew has proposed a Model_D, and extension beyond using Catastrophe Theory as a metaphor remains at the level of plausibility, not demonstration.

In part, the difficulty is conceptual and stems from reifying the society as an entity that responds to forces acting upon it, much as a physical object responds in its movements to forces acting upon it. For the physical object, the effects of forces on motion are well known and a particular situation can, in principle, be examined through the appropriate application of mathematical representation of these effects along with suitable information on boundary and initial conditions.[4] It is far from evident that a similar framework applies to whole societies. A society is composed of both material and ideational/cultural dimensions. It can only in part be modeled as a complex which is describable just in terms of material dimensions. Were the latter a sufficient characterization, it would then be reasonable to simply determine regularities that emerge from the structure constrained by boundary and initial conditions. Renfrew (1979:502-503) used this tactic when he attempted to relate the difference between a chiefdom and a state, each in the same ecological conditions, to "the *extent* to which relations within the society are still determined by kin relationships" (emphasis added); i.e., to the initial condition as given in the specific link between societal relations and kin relationships. But if kin relationships are part of the critical difference, then one must understand the culturally defined, conceptual framework within which these relations are expressed; namely the kinship terminology and its implementation through marriage rules. A change in marriage rules, for example, is not the equivalent of a change in parameter value but involves a restructuring of the whole system -- a topic that is not modeled by Catastrophe Theory.

Even more, kinship terminologies and marriage rules do not operate deterministically, but can be acted upon and manipulated for individual or subgroup ends. The kinship terminology can be mathematically modeled as an abstract structure whose properties derive from an internal logic (see Read 1984). That the terminology has structure given by an internal logic argues against viewing culture as merely a reflection of behavior whose determinants lie outside of the ideational realm. If so, explanation is not "bottom up" from externally directed behavior to internal coding as culture, but must involve the conceptual domain as both giving meaning to external events and phenomena and serving to define a framework for behavior constrained by external conditions. The linkage between conceptual structure and behavior is, evidently, complex and non-deterministic, yet constrained by external conditions. It clearly has aspects open to manipulation by individuals or subgroups for achieving both public and private goals, but such manipulation is also constrained by publicly accepted conceptual strictures such as a kinship terminology, marriage rules, and the like. This self-evaluation capacity, coupled with the ability of the actors in the situation to affect the societal means of reproduction, including both material and ideational dimensions, makes modeling of societies difficult and hard to reduce, assuming it is possible, to deterministic models.

When models incorporate parameters whose values are fixed, a non-reflective system that does not incorporate self-modification is implicitly presumed. For example, while the equation,

$$dp/dt = kp, \qquad (12)$$

is a reasonable idealization for population growth under unconstrained conditions, it implies exponential growth that cannot be maintained over long periods of time. So one postulates

$$dp/dt = kp(1 - kp) = kp - (kp)^2, \tag{13}$$

as a Model_D in which are incorporated, in summary fashion, all constraints on population size. Problematic, however, is the assumption that these constraints are assumed to be fixed. Over time scales appropriate to archaeological data these constraints are not fixed. The issue is, then, not so much one of modeling population growth under constrained conditions but modeling the dynamics of how those constraints come about and how they come to be modified. Without identifying the process by which change takes place in the birth rate, for example, Equation (13) is only descriptive, hence a Model_D and not a Model_T.

A Model_T for a stabilized population has bees given by Read (1987) for the *!Kung San*. The process driving the system towards a stable equilibrium is identified as the decision a woman may make regarding spacing of children in accordance with greater value being placed on family well being than family size, per se. The combined effects of

(1) growth rate decreasing monotonously with bath spacing,
(2) population increase leading to a declining marginal rate of return in foraging labor,
(3) allocation of limited time and energy in accordance with family well-being,

are sufficient to argue that population sire will be driven to a stable equilibrium value solely on the basis of individual self-interest (see Read 1987 for details). Interestingly, the model implies, perhaps counter-intuitively, that the "distance" between equilibrium population size and potential carrying capacity -- in the sense of the maximum population size that could be sustained over the long run without technological/structural change brat utilizing all available food resources -- should, keeping fixed the technological level of resource procurement system, increase with declining resource density since the marginal rate of return on foraging labor should decline non-linearly and proportionally more rapidly with decreasing resource density. In other words, populations whose size is stabilized by a process similar to the one advanced for the !*Kung San* should have less risk in resource poor environments and greater risk in resource rich environments, assuming the variance in the quantity of available resources through time is approximately the same in both cases.

Yet in other conditions individual self-interest may lead to run away population growth. In labor intensive agricultural systems it may be the case that children provide inexpensive labor, hence individual self-interest may lead to placing high value on pregnancy, per se. So long as the consequences of a growing population can be distributed to other sectors, such as urban centers through rural-urban migration, the large scale, societal effects of individual decision making do not feed back in the same manner and in fact the contrary may be true; a growing population may lead to agricultural intensification, hence to increased demand for cheap labor, which may be met by increased number of offspring.

GROUP AND INDIVIDUAL LEVELS

The dynamics between individual action - -which, ultimately, is the source for societal attributes measured at a more summary level -- and group properties, including societal organization and cultural systems, is a constant problem that archaeological theorizing has not adequately addressed (Keene 1983, 1985; but see Reynolds and Zeigler 1979; Johnson 1982; and Reynolds 1984 for formal accounts that address this issue). The data being analyzed -- the material remains found at sites -- are summary information, yet if explanation requires process defined at the level of individual action as in the *!Kung San* model for population equilibrium, then analysis needs to address the linkage between the two levels.

When the linkage is considered, it is often simplified by assuming culture-free properties, such as maximization of a utility function or minimization of a cost function, without taking into account the dynamics involved for the optimal solutions to be known to the individual. Or, the net effect of individual decision making is presumed to lead to an optimal solution defined at the level of the system. Hence it is assumed that the dynamics of individual decision making and how these dynamics affect global properties can be ignored. Even when the dynamics are not ignored, it is sometimes assumed that those past societies that survived sufficiently long to leave a recoverable archaeological record must be the ones that found optimal, or near optimal solutions, so that the dynamics and implications of individual decision making can be ignored.

Linear programming, in its archaeological application as a means to account for the resource mix utilized by a group of persons (e.g., Reidhead 1981; Keene 1979, 1981; Boyle 1986 -- see review by Reidhead 1979), exemplifies these difficulties most clearly (see Keene 1985 for a critique of linear programming applications in archaeology). The basic assumption is that a group of persons utilize a set R = $\{r_i\}$ of n resources, $r_i \leq i \leq n$, in such a manner that the total procurement cost, C, to obtain a basic diet, and possibly other goods, is minimized. Formally, the problem may be stated as follows. Let the cost function $C = C(r_1, r_2, ..., r_n)$ be given by:

$$C = c_1r_1 + c_2r_2 + ... + c_nr_n, \tag{14}$$

where: c_i is the cost per unit procured of the i^{th} resource.

Suppose there are constraints

$$Q_j \pounds a_{1j}r_1+a_{2j}r_2+ ... +a_{nj}r_n,\ 1 \leq j \leq m, \tag{15}$$

where: a_{ij} is the amount of the j^{th} good obtained per unit from the i^{th} resource, and

$$0 \leq r_i,\ 1 \leq i \leq n. \tag{16}$$

The constraints Q_j might represent the minimum quantity of a substance, j, such as a nutrient obtained from a resource, or a utilized part of an animal such as its skin, required by the group over some time period. The constraints, $0 \leq r_i$, limit possible solutions to non-negative quantities for each resource. The solution is found by determining the values for $r_1, r_2,..., r_n$, that minimize C and satisfy the constraints given in Equations (15) and (16) (see Strang (1986) for a discussion of the logic of linear and non-linear programming methods).

A detailed application of linear programming as applied to Netsilik Eskimo hunting practices has been given by Keene (1979), with results that "appear largely congruent with the ethnographic and historic accounts" (p. 388). Nonetheless, as Keene (1985) has discussed, there are implicit assumptions underlying the linear program model in its application to hunting and gathering economics that may not be valid. Two such assumptions that relate directly to the mathematical specification of a model will be discussed here. These are: (1) no discontinuities are allowed in the cost function and (2) it is assumed that costs are fixed per unit, i.e., that there is a fixed marginal cost.

The first problem refers to the implicit assumption that the cost per unit decreases to zero as the quantity procured decreases to zero when a cost function in the form

$$C_i = c_ir_i \tag{17}$$

is used for the i^{th} resource, r_i. However, Zechenter (1988) has pointed out that costs are often discontinuous and may include an initial cost that must be borne even when only a single unit is obtained, such as travel cost of going to a resource location. Averaging these costs over the resources procured does not eliminate the incongruity and may introduce significant distortions, as discussed below. Zechenter suggests that a cost function of the form:

$$C_i = \begin{cases} 0 & \text{if } r_i = 0 \\ c_i r_i + C_{i0} & \text{if } r_i > 0 \end{cases} \tag{18}$$

where: C_{i0} is a fixed constant specific to the i^{th} resource,
may be more realistic. Zechenter observes that a variant on linear programming known as integer programming (see, for example, Schrijver 1986) can incorporate discontinuities of this hind when finding a solution. integer programming allows the cost function to incorporate resource specific initial costs by rewriting Equation (14) in the form

$$C = c_1r_1 + c_2r_2 + \ldots + c_nr_n + C_{10}y_1 + C_{20}y_2 + \ldots + C_{k0}y_k, \tag{19}$$

where: $y_i \, \varepsilon \, \{0,1\}$.

An optimal solution now consists of values for both the r_i and the y_i that minimize C. Since the y_i are either 0 or 1, these values define which resources are part of the solution set. Other configurations are also possible: e.g., some resources may share a common initial cost or have alternative means of procurement with different initial costs, it may be that some resources must be procured in some minimum amount if at all, diet diversity may be a requirement, and so on. Resource specific specifications such as these can be incorporated by suitable modification of Equation (19) and/or by modification of the constraints given in Equations (15) and (16) (Zechenter 1988). Zechenter considers a variety of alternative scenarios for ethnographic data on resource procurement costs extrapolated to the preceramic and initial periods of coastal Peru. She concludes that during the early preceramic period, resource procurement would have focused on terrestrial resources in the coastal area, with marine resources serving primarily as supplementary resources, while the initial period would have focused on agricultural resources grown in inland valleys.

The second problem relates to the broader issue of what are the constant and variable aspects of a system. The linear programming model given in Equations (14) - (16) is static in the sense that the equations and their parameters are assumed to be fixed. The latter is equivalent to asserting that resources have a fixed marginal cost, regardless of level of procurement -- hardly a realistic assumption for resource procurement by hunting and gathering societies. Keene (1979) is aware of the problem for he comments that the assumption of a fixed cost per unit regardless of quantity obtained (see Equation (14)) is unrealistic (see also Keene 1985). To resolve the difficulty Keene used an *average* cost (p. 378). But an average cost is not the same as the actual cost of obtaining an item, and the difficulty of using an average for the actual cost can also be seen in the data used by Keene. For example, Keene notes that whereas the predicted diet would never include polar bear because of its high average cost due to the cost of deliberately searching for polar bears, nonetheless one would expect it to be hunted if it were accidentally encountered (p. 391). In other words, restructuring costs as average costs introduces distortions (Reidhead 1979:561). Further, average costs cannot be extrapolated to different conditions, such as larger or smaller populations, in order to determine how diet composition might change as populations expand or decrease. Change in population size would be expressed through a multiplicative factor applied to Equation (15) and hence would not affect the solution found.[5]

An alternative to assuming fixed marginal costs for all resources is to estimate the changing marginal cost for each resource as the amount procured increases. Christenson (1981) followed this procedure when he examined changing resource mix in archaeological sites ranging in time from paleoIndian to historic in the Midwest. He comments that "The observed trends are close to the predicted trends [and] ... indicate a growing population diversifying subsistence by adding more and more costly food items to feed itself from a fixed resource base" (p. xv).

The optimal resource mix is defined to be one for which the marginal cost of each resource is the same, given a total quantity satisfactory for the needs of the group. More formally, for the set R = $\{ r_i \}$ of resources, let the cost function for the i^{th} resource be given by

$$C_i = C_i(n_i), \tag{20}$$

where: n_i is the quantity procured for the i^{th} resource.

The marginal cost for the i^{th} resource is given by the derivative of the cost function, dC_i / dn_i. Assume that the second derivative is positive ($d^2C_i / dn_i^2 \geq 0$), so that the marginal cost curve is concave upwards, that is, marginal costs increase with increasing n_i. The optimal resource mix occurs when the minimum quantity required, N, is met with resources all of which have the same marginal cost, and at the least total cost; that is, the set of resources with indexes in the index set $I = \{i_1, i_2, \ldots, i_k\}$ will be an optimal solution if

$$N = \sum_{i\, e\, I} n_i \tag{21}$$

and

$$dC_i / dn_i = dC_j / dn_j, \text{ for all } i, j\ e\ I, \tag{22}$$

subject to the condition that

$$C = \sum_{i\, e\, I} C_i(n_i) \tag{23}$$

has been minimized. In other words, the resource set indexed by I is an optimal solution if no other set, J, of resources satisfies equations (21) and (22) with smaller total cost, C, than that given by Equation (23) for the resource set indexed by I.

The rationale for defining the optimal resource mix in this manner is straightforward. If the marginal cost for resource r, whether or not it is currently part of the resource mix, is less than the marginal cost for resource s currently part of the resource mix, then 1 unit of resource r may be substituted for 1 unit of resource s, thereby keeping fixed the total quantity, N, yet reducing the total cost, C. In this context, a "unit" does not refer to the resource as a whole, but to some arbitrary unit such as 1000 Cal. Equilibrium in this substitution process will occur when all resources in the resource set, I, have equal marginal costs that are also less than the marginal cost of any resource not in the set I. An implicit assumption is that the difference in marginal costs is known to the actors in the situation. The assumption is not overly restrictive since the marginal costs refer to the costs accrued when a unit of resource is obtained and presumably the actors procuring resources are aware of the costs, such as energy expended, involved.

Note that the solution set, I, is a function of N, the total quantity procured, since marginal cost curves may not be congruent. Hence, according to this model, an optimal resource mix is not derived from just knowing the procurement costs of resources, but must also take into account the way in which procurement costs may vary with the total quantity, N, that is required. That procurement costs are not fixed for a given resource is excluded from the solution found not only in linear programming modes, but in other frameworks as well, such as optimal foraging theory where arguments often ignore the fact that costs depend on the intensity with which a resource is exploited.

While Christenson assumed the marginal cost for a resource increases with the quantity obtained (see also Keene 1979), the definition of the optimal mix given above is not restricted by the form of the marginal cost curves, although with some curves there might be more than one solution. Linearity assumptions, continuity in costs, and so on, are no longer part of the analytical solution. While application to a specific context depends upon estimation of the marginal cost curves -- which may be a difficult task -- reasonable qualitative assessments may be possible even with limited data as Christenson (1981) demonstrated in his study.

Though modeling based on marginal costs relaxes the more stringent assumptions underlying linear programming, other assumptions, which are also part of linear programming, are still necessary. Two of these are: (1) substitutability of resources and (2) constancy of parameters. The first assumption refers to the fact that it is assumed that one unit of resource *r* may replace one unit of resource *s*. Substitutability may, ostensibly, be embedded into the model through measuring all resources via some common currency such as calories. But to do so presumes that all resources were equivalent within the cultural system in question. Resources are not just a means to provide caloric or other nutritional requirements, but are heavily endowed with cultural meanings that embed them into a variety of different contexts, each of which will affect choices made about resources to be procured (Jochim 1983). For example, the fact that vegetal foods arse generally gathered by women and animals hunted by men implies that the one kind of food is not equivalent to the other kind. Typically, meat is redistributed according to various cultural rules regarding who has rights with respect to hunted animals, whereas gathered foods are generally under the control of the women who has gathered them. Among the *!kung San*, a man's obligations to his parent-in-laws were satisfied through providing meat he hunted, not through gathered foods obtained by his wife (Marshall 1976). Thus, while the two kinds of foods may be nutritionally substitutable in terms of calories, the same is not true at the cultural level. But assessing just how much of an impact the cultural side of the equation has on actual practice is problematic and leads into a topic which is not a part of this review.

The second assumption brings us back to an earlier theme, the presumption that the structural form and parameters of models are constant. The marginal cost model, while it allows for a changing resource mix according to changing total demand, assumes constancy in the cost of procuring. processing and preparing of resources and how these costs are affected by social organization and level of technology. Yet change in the items considered as food resources, innovation in technology and restructuring of social organization as these affect obtaining and utilizing resources are precisely some of the major questions addressed by archaeologists. At the same time, these are aspects that are generally assumed to be unchanging in most applications of mathematical models.

There is a paucity of models aimed at formally modeling the process of change, though some of the chapters in the book *Transformations: Mathematical Approaches to Cultural Change* (Renfrew and Cooke 1979) are notable exceptions. The lack may be due, in part, to borrowing models from other disciplines where change is of less concern than it is to anthropologists, in general, and archaeologists, in particular. In part, it may be due to the fact that formally modeling change is difficult, and the more traditional modeling tools that have been developed in mathematics are not always well suited for the indeterminism and uncertainty that is associated with the processes by which systems change their structure. Dynamic structural modeling is a powerful framework for analyzing the properties of systems, but does not, in and of itself, provide the means for analyzing the properties of systems that are self-reflective and capable of both affecting and defining how they are going to change, as is true of human systems. While human systems are constrained by external conditions, nonetheless they have a level of internal organization -- what we call culture and social organization -- that is, itself. the product of the human components of the system. Perhaps because culture, except in its material products, is riot directly observable in archaeological data, and perhaps because the things observable are directly the result of individual behavior, there has been much emphasis on purported "laws" of behavior as the foundation for the explanatory arguments that archaeologists are trying to develop. This, I argue, is not likely to succeed.

To the extent that there *are* "laws" affecting human behavior, they must be due to properties of the mind that are the consequence of selection acting on genetic information. As a consequence, "laws" of behavior are inevitably of a different character than laws of physics such as $F = ma$. The latter, apparently, is fundamental to the universe itself; behavioral "laws" such as 'rational decision making' are true only to the extent to which there has been selection for a mind that processes and acts upon information in this manner. It is not an external property that has prior existence, but exists -- if at all -- only because the mind is constricted so that sensory information is processed in a way that leads to actions which we call 'rational'. But given the complexity of the human mind, it is far from evident that there is any simple linkage between genetic information and the manner in which the human mind processes information. Without virtually isomorphic mapping from genetic information to properties of the mind, searching for universal laws of behavior as a means to develop explanatory models of humans systems in analogy with the role that physical laws have played in physics in developing explanatory models of the universe is a chimera. It is not so much laws of behavior that we need, but Model_T models of how complex, information processing, self-reflective, self-restructuring systems operate, develop and change (e.g., Reynolds and Zeigler 1979; Johnson 1982; Reynolds 1984; Mithen 1987). That there may be commonality in how different groups facing similar circumstances find solutions is not being questioned; only the emphasis on behavioral "laws" as the *foundation* upon which explanatory arguments must ultimately rest. Mathematical modeling aimed at Model_T kinds of models can be no better than the processes which are being expressed formally. If the assumed processes are faulty, then it is equally the case that models built upon those assumptions are faulty when application is made to the empirical context.

CONCLUSION

In the first part of this review I have argued that there are two different, but related kinds of models relevant to explanatory arguments: Model_D models aimed at expressing idealization of empirical conditions and Model_T models aimed at expressing theoretically and abstractly defined relationships. These two modalities do not, of course, exhaust the full sense in which models have been used in archaeology. Voorrips (1987) discusses four types of models that have been used in archaeology, two of which correspond to the Model_T and Model_D given here. The distinction relates to both the domain for which the model is constructed -- the empirical versus the theoretical domain -- and the nature of confirmation -- empirical test versus logical consistency with posited relationships. Different methods are typically employed as well. Statistical methods are paramount with Model_D models as the goal is to establish congruency between model and data. whereas symbolic, logical arguments are paramount with Model_T models as the goal is to express the deduced consequences of relationships defined abstractly. Neither kind of modeling has priority as scientific arguments involve both modalities. Scientific explanation, in this framework, can be viewed as demonstrating isomorphism between the two kinds of models for the situation in question. For there to be explanation, there must be both valid idealization of real world phenomena to isolate those properties for which explanation is desired, and there must be an embedding of the relationships found in the idealized data into a suitable theory in order to demonstrate the relationships as a deducible consequence given the structuring properties around which the theory is constructed. In terms of practice, however, most "mathematical" modeling in archaeology has utilized $Model_D$ constructs (Voorrips 1987; Cowgill 1955) and less systematic work has been done on $Model_T$ constructs. As Cowgill (1986:371) has commented: "[archaeological] theory itself is rarely couched in mathematical terms".

Mathematical formalism applied to archaeological concepts as an end in itself is a sterile exercise. Mathematical/formal representation aimed at providing a symbolic framework wherein the logic of principles said to be structuring a domain cam be examined and implications determined is, however, a means to extend archaeological reasoning to more subtle. and less obvious properties of those principles. When there is concordance between concept and representation, the result can only be a fuller and deeper understanding of the phenomena under the purview of the archaeologist. This is the promise held out by the application of mathematical constructs to archeological problems.

NOTES

1. The first part of this review is a revised version of a paper given at the symposium, Theoretical Frameworks for the use of Mathematical Methods in Archaeology, held as part of the eleventh Congress of the *Union Internationale des Sciences Préhistoriques et Protohistoriques,* August 31 - September 5, 1987. Funds for attending the Congress were provided by the UCLA Academic Senate.

2. The review makes no attempt to be exhaustive of studies that have applied mathematical formalism in the sense of extending archaeological reasoning. In particular, recent work on analyzing decorative patterns using pattern mathematics (e.g., Zaslow and Dittert 1977; Washburn and Crowe 1988) and artifact design based on formal grammars (e.g., Read 1986) will not be considered here. In addition, earlier examples of the application of mathematical formalism (e.g. Read 1974; Zubrow 1975) are not considered.

3. Since the term 'successor' is a primitive, it has no defnition in the axiomatic system. Hence any definition consistent with the axioms is permissible. The intuitive sense in which it is usually taken corresponds to the idea that any set of objects has associated with it a quantity representing the number of things in the set. If a set is augmented by one object, then its new quantity would be the successor to the quantity associated with the set prior to its augmentation (see also Read 1987).

4. Or, at least complete characterization can be made for linear dynamical systems. Non-linear systems are more complex and difference equations can exhibit "chaotic" behavior even when deterministic. Trajectories for the nonlinear system may be heavily affected by minor perturbations, hence making prediction of future states from the structural equations and initial conditions problematic. See Strang (1986:471-507) for a comparative discussion of linear, nonlinear and discrete systems.

5. Linear programming does take into account marginal costs through what are called shadow prices (see discussion by Keene 1979, 1985, and Reidhead 1981), but shadow prices are less of a global consideration of marginal cost than an analysis of the local effects of marginal cost on the optimal solution.

6. There are notable exceptions such as the Tiwi who also included certain animals as part of the resources obtained by women (Goodale 1971).

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